이학박사 학위논문

Many-Body Condensate Fragmentation in a Single Trap: Theory of Detection and Phase State Characterization

단일 덫 다체계 응집상태에서의 파편화:
측정 및 위상 상태를 이용한 표현에의 이론적 접근

2016년 2월

서울대학교 대학원

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Abstract

Dilute atomic gas below the critical temperature has been a field of interest due to its long-range, large coherence among over the whole system. Also parameters of the atomic gas system can easily be engineered with less unwanted external disturbance. And Bose-Einstein Condensate (BEC) is described by one macroscopically occupied mode (macroscopic mode) as a mean field together with small quantum fluctuations.

In this thesis, a fragmented state which has two or more macroscopic mode in bosonic system, is investigated to find whether there exists interesting effects from correlation between macroscopic modes. Firstly, definition of fragmented state is introduced and further classification is presented to exclude trivial cases such as fragmented state appearing in a double well. Emergence of fragmented state is claimed as a ground state transition from BEC of the system in zero temperature limit.

Macroscopic modes cannot be simply dealt with density matrix in terms of momentum eigenstates as in homogeneous thermal gas. Hence it is impossible to get the exact variational calculation considering spatial orbitals of each modes and occupation numbers without enough specification of system and constraints. In this thesis we consider a quasi-1d gas in a single inhomogeneous but symmetric (e.g. harmonic) trap, in order to seek for two-mode fragmentation.

Field operator is truncated to obtain effective Hamiltonian for two macroscopic modes, which is just one step beyond the Gross-Pitaevskii (GP) equation. Then specification of the system into quasi-1d case is done by integrating out the other directions. And original macroscopic mode is considered to be e.g. Gaussian or Thomas-Fermi (TF) with even parity which are popular model describing BEC, since we consider fragmented state to stem from BEC as interaction strength increases. Fragmented state has larger single particle energy than BEC, thus odd parity of additional macroscopic mode with large overlap between the original mode in magnitude is assumed to minimize increase of single particle energy by introducing additional mode.

Fragmented state satisfying energy equation has lower energy than BEC state as repulsive interaction strength goes up from limited variational calculation. This fragmented state has significant, and negative pair coherence related to pair tunneling term. There exist almost degenerate two fragmented states. They collapse for very small tunneling perturbation into symmetric, anti-symmetric superpositions which is stable fragmented states. Spatial coherence and density-density correlation are investigated for detection of unique characteristic(s) in the fragmented state. Strong fluctuation of density-density correlation after Time-of-Flight (TOF) is discussed, and is compared to double well case which has Hanburry-Brown-Twiss (HBT) correlation.

Phase state is introduced, which is useful in analyzing interference between two independent BECs, to interpret fragmented state in a single trap further. Condition to apply phase state formalism for general two-mode state is studied, and phase state relates fragmented state to a superposition of two phase states of opposite phases. Furthermore, condition for fragmentation in terms of phase state coefficient is stated which can support the relation between fragmented state and peculiar correlation function.

Approximate coherent state is established, to further fertilize and broaden the possible interpretation on fragmented state. By comparing phase state and approximate coherent state, a clue is found which leads us to the analogy between fragmented state and photonic cat state. To identify fragmented state as cat state, superposition of approximate coherent states of opposite phases, an expression of two-mode state in terms of approximate coherent state basis is investigated. Relation between coefficients of two different types of superpositions were found by transforming generalized expression of negative pair coherent (NPC) state including fragmented state, into a superposition of approximate coherent states. Further, direct relation between fluctuation in density-density correlation and quadrature fluctuation is discussed.

Keywords: Cold atomic gas of boson, Fragmentation, Single trap, Density-density correlation, Phase state, Approximate coherent state, Negative pair coherence, Cat state, Quadrature fluctuation

Student Number: 2010-20356

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Chapter 1

Introduction

After first Bose-Einstein Condensate (BEC) was realized in experiments at mid of 1990 [1, 2], atomic gas at ultracold temperature T of nK scale has been one of interesting research area both in experiment and theory. The power of cold atomic gas comes from the fact that

- Number of considerable factors or parameters are few and already well explained in theory.
- Thus theory, or model Hamiltonian, correctly describes the system hence theory and experiment are closely related.
- Easy to engineer details or characteristics of corresponding theory therefore can simulate quantum phenomena from model Hamiltonian.

Atomic gas can be classified as fermionic case and bosonic case, depending on angular momentum number, and this thesis deals with bosonic case only. BEC appears as temperature goes down under certain critical temperature T_c of which order of magnitude determined from

$$\lambda_{T_c} \sim d \tag{1.1}$$

where d is interparticle spacing and

$$\lambda_T = \frac{h}{\sqrt{2\pi m k_B T}} \tag{1.2}$$

is thermal De-Broglie length which is De-Broglie wavelength for corresponding kinetic energy from thermal motion. Under T_c, $\mathcal{O}(N)$ macroscopic number of particles starts to occupy one energy level where N is total particle number of the given system. This is not the case following conventional Bose-Einstein statistics of ideal gas above T_c where occupation number n_i comes from

$$n_i(\epsilon_i) = \frac{g_i}{e^{(\epsilon_i - \mu)/kT} - 1} \tag{1.3}$$

with ϵ_i is single particle energy without interaction and μ is chemical potential. One thing to be stressed is that under this circumstances, $\mathcal{O}(N)$ particles gathers into one energy level, or mode. It is straightforward in non interacting case, but not so trivial if there exists interaction.

There are some pedagogical references and textbooks about BEC and quantum gas e.g. [3, 4, 5]. Thus in this thesis BEC and related Gross-Pitaevskii (GP) equation, Bogoliubov approximation, etc. will not be dealt or only small amount of pages are devoted. Rather, here we want to illustrate what ground state of cold atomic gas system will be there beyond BEC as we increase interaction strength while temperature is low enough so that $T \to 0$ limit is valid to describe condensate part, occupying one or more mode with $\mathcal{O}(N)$ particles.

To briefly explain about BEC under interaction, let's with general many-body Hamiltonian

$$\begin{aligned} \hat{H} &= \int d\mathbf{r}\, \hat{\psi}^\dagger(\mathbf{r})\, \hat{h}\, \hat{\psi}(\mathbf{r}) + \frac{1}{2} \iint d\mathbf{r} d\mathbf{r}'\, \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}')\, \hat{V}(\mathbf{r}, \mathbf{r}')\, \hat{\psi}(\mathbf{r}') \hat{\psi}(\mathbf{r}) \\ \hat{h} &= \hat{K} + \hat{U}. \end{aligned} \tag{1.4}$$

Where $\hat{K} = -\hbar^2 \nabla^2 / 2m$ represents kinetic energy and $\hat{U} = U(\mathbf{r})$ represents potential energy. Now introducing field operator expansion

$$\hat{\psi}(\mathbf{r}) = \sum_{0}^{\infty} \langle \mathbf{r} | i \rangle \hat{a}_i \tag{1.5}$$

and taking only one mode $\hat{a}_0$ yields $\hat{\psi}(\mathbf{r}) \simeq \psi_0(\mathbf{r})$ neglecting commutation relation $[\hat{a}_0, \hat{a}_0^\dagger] = 1$. For contact interaction $V(\mathbf{r}, \mathbf{r}') = \frac{g}{2} \delta(\mathbf{r} - \mathbf{r}')$,

$$\hat{H} = \int d\mathbf{r}\, \psi^*(\mathbf{r})\, \hat{h}\, \psi(\mathbf{r}) + \frac{1}{2} \iint d\mathbf{r} d\mathbf{r}'\, \psi^*(\mathbf{r}) \psi^*(\mathbf{r}')\, \hat{V}(\mathbf{r}, \mathbf{r}')\, \psi(\mathbf{r}') \psi(\mathbf{r}) \tag{1.6}$$

Taking function derivative with respect to $\psi(\mathbf{r})$, one gets equation satisfied by

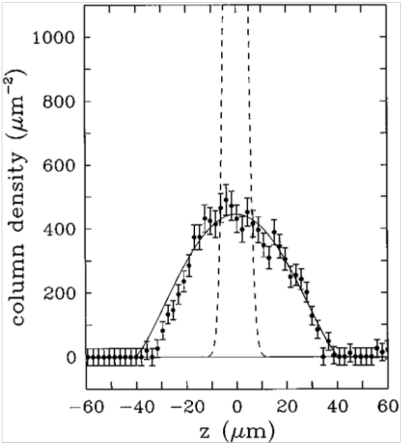

Figure 1.1: Column density profile of BEC in a harmonic trap (solid) [6]. Dashed line indicates expected density profile without interaction.

ground state wavefunction, or orbital, $\psi(\mathbf{r})$; GP equation.

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r})\right)\psi(\mathbf{r}) + g|\psi(\mathbf{r})|^2\psi(\mathbf{r}) = \mu\psi(\mathbf{r}) \tag{1.7}$$

One characteristic of interacting BEC is that, though interaction strength is so weak and atomic gas is so dilute that s-wave scattering wavelength a_s is smaller than interparticle spacing d, interaction plays crucial role. Following Fig.1.1 is measured column density of BEC. This well visualizes how small interaction strength dramatically changes characteristics of BEC with interaction from those of ideal gas BEC case.

Let us briefly talk about known consequence of further increasing interaction strength, in quasi-1d case. This can illustrate how and in which direction research contained in this thesis was lead. Here we consider repulsive interaction, where increasing attractive interaction anyway will bring collapse of atomic gas. Especially as we consider finite N case, appearance of one 'additional' mode is likely to happen [7]. In addition, for example when we consider 2D or 3D, it is more likely to have more than one equally probable 'additional' $\mathcal{O}(N)$ occupied modes due to isotropy (If isotropy is severely harmed and certain direction is frozen, it is equivalent to reduction of dimension). As can be seen from above two arguments, quasi-1d case is favorable candidate to have fragmented state as new ground state and it is. Also, the fragmented state in a single trap, of

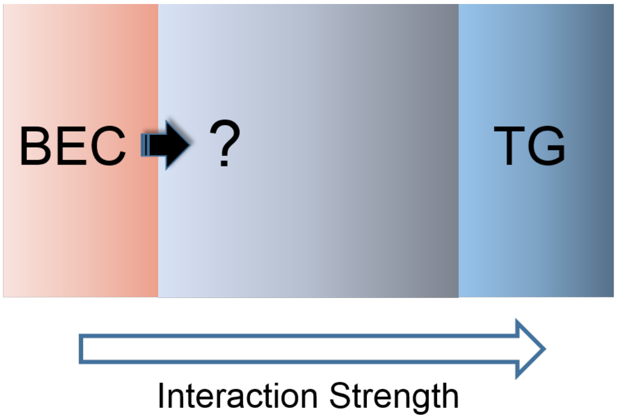

Figure 1.2: Schematic phase diagram of quasi-1d quantum gas.

interest in this thesis, less likely to appear in higher dimension [8, 9].

Basically quasi-1d cold atomic gas is a subject of interest in this thesis, and how fragmented state appears as ground state and what that many-body state tells us is major concern. Before doing so, Let us briefly explain the concept and definition of fragmented state.

Chapter 2

Fragmented State

In this chapter, we first introduce single particle density matrix and definition of fragmented state; many-body state with more than one $\mathcal{O}(N)$ occupied modes. And the reason why fragmented state can be interesting subject is to be studied. Also it is stated that in which basic thinking our research on fragmented state in this thesis has been carried out. As rather trivial example, fragmented state in a double well is introduced and characteristic is discussed.

2.1 Definition of Fragmented State

As illustrated in short explanation on BEC in previous chapter, $\mathcal{O}(N)$ occupation at certain mode infers there exists something beyond conventional Bose-Einstein statistics. And more than one $\mathcal{O}(N)$ occupation can happen as interaction strength increases right beyond BEC regime under $T \to 0$ limit. Thus counting occupation number for each mode is one meaningful way to classify bosonic many-body states. Single particle density matrix (SPDM) $\langle \hat{a}_i^\dagger \hat{a}_j \rangle$ can be always diagonalized in certain basis; $\langle \hat{a}_i^\dagger \hat{a}_j \rangle = n_i \delta_{ij}$ with uniquely defined occupation number n_i.

Thermal gas and BEC can be defined by examining $n_0 \sim \mathcal{O}(N)$ or not where n_0 is largest among n_i. Going further, choosing specific basis with ordering $n_0 \geq n_1 \geq n_2 \cdots$, a many-body state is called as 'fragmented state' when not only n_0, but also n_1 (and possibly $n_2, n_3, \cdots$) is $\mathcal{O}(N)$ [10]. Let's call a mode i with $n_i = N_i \sim \mathcal{O}(N)$ as 'macroscopic mode' with notation capital N_i introduced to make distinction from non-macroscopic modes. For two macroscopic modes, one can define degree of fragmentation $\mathcal{F}$ as

$$\mathcal{F} = 1 - \frac{|N_0 - N_1|}{N} \tag{2.1}$$

which quantifies how much the state is fragmented. To understand what kinds of potential usefulnesses are contained in this fragmented state which is many macroscopic modes case, let us get back to BEC case. BEC, one macroscopic mode case, shows unique features such as superfluidity, vortex, or highly peaked momentum distribution out of Boltzmann distribution originated from that one macroscopic mode. In ideal gas case, total cold atomic gas is composed of condensation part and thermal gas part, thermal gas part is often considered as simple background.

In this context, most of interesting features of interacting BEC can be captured by simple model with brutal approximation $\hat{\psi}(\mathbf{r}) \simeq \psi(\mathbf{r})$ breaking number conservation which is allowed for large particle number N[11]. Or up to $\hat{\psi}(\mathbf{r}) \simeq \psi(\mathbf{r}) + \delta\hat{\psi}(\mathbf{r})$, while $\delta\hat{\psi}(\mathbf{r})$ expresses small fluctuation or depletion. In fragmented state, however, $\hat{\psi}(\mathbf{r})$ cannot be expressed as sum of mean field which is simple c-number and small fluctuation. In other word, for BEC $\hat{a}_0, \hat{a}_0^\dagger \sim \sqrt{N_0}$ is generally true but for fragmented state simple approximation $\hat{a}_0 \sim \sqrt{N}_0, \hat{a}_1 \sim \sqrt{N}_1$ does not work. And this leads to possibilities of novel quantum phenomena from relation(s) between macroscopic modes.

In summary, (bosonic) many-body state of interest here is

1. Fragmented state with two or more macroscopic occupation N_i
2. Creation and annihilation operator $\hat{a}_i$ cannot be approximated as c number $\sqrt{N_i}$
3. Ground state of cold atom system, therefore realizable as condensation part at ultracold atom experiment.

And this thesis introduces one of fragmented state satisfying conditions especially, in a single trap geometry, as possible ground state of many-body system and is initiated to find reasonable interpretation on that fragmented state from various aspects. As will be shown in following example, finding such fragmented state is not trivial.

2.2 Finite N Issue

It is worthwhile to discuss following subtle issue. Above definition of macroscopic mode and fragmented state is based on concept of $\mathcal{O}(N)$ which only holds for thermodynamic limit $N \to \infty$ in principle. However, particle number is always finite and is about $10^2 \sim 10^7$ for condensation part in typical cold atomic gas. Hence sometimes definition of $\mathcal{O}(N)$ is vague, e.g. when $0.1N = \sqrt{N}$ for $N = 10^2$. Therefore definition of 'macroscopic' or $\mathcal{O}(N)$, and also definition of fragmented state can be subjective in some sense.

Firstly, since we assumed additional mode(s) in condensation part not to be neither fluctuation part $\delta\hat{\psi}(\mathbf{r})$ representing quantum depletion nor under conventional Bose-Einstein statistics. This can be one standard to discriminate mode constituting condensation part in fragmented state from other modes. Also, recalling the case of BEC, an existence of BEC was confirmed as unique property of BEC, e.g. highly peaked momentum distribution [2], was revealed in experiment. In the end, confirmation of an existence of fragmented state and additional macroscopic mode(s) depends on whether unique property of that fragmented state can be drawn out from an experiment, since occupation number might not be accessible in the experiment.

2.3 Fragmented State in a Double Well

Let us introduce one example of fragmented state. Considering cold atomic gas in a symmetric double well with short range repulsive interaction and high barrier yields trivial example of the fragmented state as ground state in temperature $T \to 0$ limit. Letting $\hat{a}_L^\dagger$, $\hat{a}_R^\dagger$ each be creation operator creating one particle at left (L) well and right (R) well, ground state $|\mathrm{DW}\rangle$ becomes

$$|\mathrm{DW}\rangle = \frac{(\hat{a}_L^\dagger)^{N/2}(\hat{a}_R^\dagger)^{N/2}}{\sqrt{(N/2)!(N/2)!}}|0\rangle \tag{2.2}$$

equally populating both wells, $N_L = N_R = N/2$ with definite particle numbers at each well.

This state minimizes repulsive interaction at each well where interaction between left and right wells are small due to short interaction range. And tun-

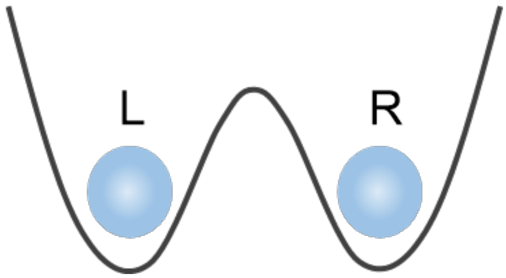

Figure 2.1: Schematic picture of fragmented state in a double well

neling between two sites, often represented as e.g. $\hat{a}_L^\dagger \hat{a}_R$, is suppressed due to high barrier between two wells. Since

$$\langle \hat{a}_L^\dagger \hat{a}_R \rangle = \langle \hat{a}_L^\dagger \hat{a}_R \rangle = \frac{N}{2}, \; \langle \hat{a}_L^\dagger \hat{a}_R \rangle = 0 \tag{2.3}$$

$|\mathrm{DW}\rangle$ is fragmented state with two macroscopically occupied modes $\psi_L(\mathbf{r}) = [\hat{\psi}(\mathbf{r}), \hat{a}_L], \;\psi_R(\mathbf{r}) = [\hat{\psi}(\mathbf{r}), \hat{a}_R]$ and is also ground state of given many-body system.

However, for $|\mathrm{DW}\rangle$ it is possible to approximate $\hat{a}_L \simeq \sqrt{N_L}$, $\hat{a}_R \simeq \sqrt{N_R}$, and $|\mathrm{DW}\rangle$ is nothing but two independent BECs spatially separated. This can be shown from following argument. Acting $\hat{N}_L = \hat{a}_L^\dagger \hat{a}_L, \hat{N}_R = \hat{a}_R^\dagger \hat{a}_R$ to $|\mathrm{DW}\rangle$ yield

$$\hat{N}_L \, |\mathrm{DW}\rangle = \hat{N}_R \, |\mathrm{DW}\rangle = \frac{N}{2} \, |\mathrm{DW}\rangle \,, \; \langle \hat{a}_L^\dagger \hat{a}_R \rangle = 0. \tag{2.4}$$

Since correlation functions such as $\left\langle \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r}')\right\rangle$ are number conserving, equal number of creation operators and annihilation operators within, from following field operator expansion

$$\hat{\psi}(\mathbf{r}) = \psi_L(\mathbf{r})\hat{a}_L + \psi_R(\mathbf{r})\hat{a}_R + \cdots \tag{2.5}$$

we can calculate any n-th order number conserving correlation function from

$$\left\langle (\hat{a}_L^\dagger)^a (\hat{a}_R^\dagger)^{n-a} (\hat{a}_L)^b (\hat{a}_R)^{n-b} \right\rangle \tag{2.6}$$

where $\psi_L(\mathbf{r}), \psi_R(\mathbf{r})$ is given. If $a = b$, it is possible to order above operator $(\hat{a}_L^\dagger)^a (\hat{a}_R^\dagger)^{n-a} (\hat{a}_L)^b (\hat{a}_R)^{n-b}$ into only function of $\hat{N}_L$ and $\hat{N}_R$. If $a \neq b$, it is possible to express $(\hat{a}_L^\dagger)^a (\hat{a}_R^\dagger)^{n-a} (\hat{a}_L)^b (\hat{a}_R)^{n-b}$ to be function of $\hat{N}_L$, $\hat{N}_R$ and

$\hat{a}_L^\dagger \hat{a}_R$ or $\hat{a}_R^\dagger \hat{a}_L$ depending on whether $a > b$ or $a < b$. $\hat{N}_L$ and $\hat{N}_R$ in bracket can be pulled out by exchanging into $N_L = N_R = N/2$ without affecting bra $\langle \mathrm{DW}|$ or ket $|\mathrm{DW}\rangle$. And this fact leads to

$$\left\langle (\hat{a}_L^\dagger)^a (\hat{a}_R^\dagger)^{n-a} (\hat{a}_L)^b (\hat{a}_R)^{n-b} \right\rangle = 0 \tag{2.7}$$

unless $a = b$ from $\left\langle \hat{a}_L^\dagger \hat{a}_R \right\rangle = 0$ and $\left\langle \hat{a}_R^\dagger \hat{a}_L \right\rangle = 0$. This argument holds for also $N_L \neq N_R$ case. In the end, every correlation function for fragmented state similar to $|\mathrm{DW}\rangle$ can be evaluated by $\hat{a}_L \simeq \sqrt{N_L}$, $\hat{a}_R \simeq \sqrt{N_R}$ except small negligible error from commutation relation $[\hat{a}_L, \hat{a}_L^\dagger] = 1$ and $[\hat{a}_R, \hat{a}_R^\dagger] = 1$. To avoid confusing this rather trivial fragmented state with the fragmented state of interest, in this thesis 'fragmented state' is not double well fragmented state unless double well is explicitly written together with fragmented state.

Chapter 3

Fragmented State in a Single Trap

In this chapter, fragmented state in a single trap appears yielding lower total energy, calculated from effective two-mode Hamiltonian, than BEC with increasing strength of repulsive interaction. This chapter has lots of arguments and not so straightforward, so let us summarize contents inside chapter 3 with few paragraphs in the following.

Fragmented state with more macroscopic modes is interesting object if correlation between macroscopic modes induces distinctive phenomena from BEC or thermal gas above T_c. When we cannot approximate $\hat{a}_0 \simeq \sqrt{N_0}, \hat{a}_1 \simeq \sqrt{N_1}, ...$, it is the case beyond mean-field description therefore which will bring us beyond 'mean-field+small quantum fluctuation' regime in cold atomic gas of boson. To have such quantum many-body state, here quasi-1d bosonic gas in a single inhomogeneous trap with repulsive contact interaction $V(\mathbf{r}, \mathbf{r}') = g/2\delta(\mathbf{r}, \mathbf{r}')$ is investigated.

Few-mode approximation is introduced from full Hamiltonian by truncating up to two macroscopic modes. Ignoring the other modes is discussed, from BEC case to general M macroscopic modes case. Though other modes can have finite occupation, it is possible to describe macroscopic modes separately when there is no strong coherence between macroscopic modes and other modes. Two-mode Hamiltonian, effective Hamiltonian beyond GP equation, is obtained from truncation.

If additional modes are considered to get new many-body ground state transition from BEC with increasing interaction strength, it is legitimate to consider one additional mode in quasi 1d case. Bringing new macroscopic mode anyway increases single particle energy composed of kinetic energy and external potential. Hence, ground state emerges right after BEC would have just one more additional modes in quasi 1D. In 2d and 3d there exists two and three additional modes can appear degenerated in single particle energy with isotropic system. After possibility of fragmented state was studied in [19], not only matter of simple description, but also it was shown that as dimension increases there is tendency of decrease of fragmentation [8, 9]. Thus quasi 1d is best candidate to find fragmentation for now.

To have ground state with two-mode, when we choose ansatz for additional mode, we would call mode 1 here, it is needed to pump up a magnitude of contact interaction strength to compensate increase of single particle energy. Thus large overlap between original mode, mode 0, is required. Assuming symmetric geometry mode 0 entails even spatial parity invariant under $z \to -z$ where z is direction of interest in quasi-1d geometry. Therefore pursuing odd parity for additional mode 1 is quite natural choice in both minimizing single particle energy and increasing overlap between mode 0 in a single trap.

Introducing even-odd parity for mode 0 and 1, Josephson tunneling between two modes vanishes, and significant pair coherence $\left\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \right\rangle$, which is negative, is obtained from energy equation. Fragmented state appears with increasing interaction strength, at least identified to have lower energy than BEC, with limited variational calculation in [8].

Actually fragmentation can vanishes from revival of pure imaginary first order coherence $\left\langle \hat{a}_0 \hat{a}_1 \right\rangle$ even with negative pair coherence for superposition of nearly degenerated fragmented states $|\text{Even}\rangle$ and $|\text{Odd}\rangle$ [21]. This degree of freedom in degeneracy, however, can be broken by small perturbation maintaining stable fragmentation unless perturbation is too strong to break down fragmented state itself [20]. Thus in static configuration, it is natural to consider fragmented state instead of state with imaginary first order coherence.

3.1 Two-Mode Approximation

We again start with following general interacting many-body Hamiltonian

$$\begin{aligned} \hat{H} &= \int d\mathbf{r}\, \hat{\psi}^\dagger(\mathbf{r})\, \hat{h}\, \hat{\psi}(\mathbf{r}) + \frac{1}{2} \iint d\mathbf{r} d\mathbf{r}'\, \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')\, \hat{V}(\mathbf{r},\mathbf{r}')\, \hat{\psi}(\mathbf{r}')\hat{\psi}(\mathbf{r}) \\ \hat{h} &= \hat{K} + \hat{U} \end{aligned} \tag{3.1}$$

where $\hat{K}, \hat{U}$ denotes for kinetic energy and potential energy respectively. A field operator $\hat{\psi}(\mathbf{r})$ is expanded in terms of any complete orthonormal basis $\{\psi_i\}$ (or $\{|i\rangle\}$ where $\psi_i(\mathbf{r}) = \langle \mathbf{r}|i\rangle$) of one particle Hilbert space as

$$\hat{\psi}(\mathbf{r}) = \sum_{i=0}^{\infty} \psi_i(\mathbf{r})\hat{a}_i, \quad [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} \tag{3.2}$$

One can write spatial coherence $\rho_1(\mathbf{r},\mathbf{r}') = \langle \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r}')\rangle$, and density-density correlation $\rho_2(\mathbf{r},\mathbf{r}') = \langle \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')\hat{\psi}(\mathbf{r}')\hat{\psi}(\mathbf{r})\rangle$ for given basis set $\{\psi_i(\mathbf{r})\}$,

$$\begin{aligned} \langle \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r}')\rangle &= \sum_{i,j} \psi_i^*(\mathbf{r})\psi_j(\mathbf{r}')\langle \hat{a}_i^\dagger \hat{a}_j\rangle \\ \langle \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')\hat{\psi}(\mathbf{r}')\hat{\psi}(\mathbf{r})\rangle &= \sum_{i,j,k,l} \psi_i^*(\mathbf{r})\psi_j^*(\mathbf{r}')\psi_k(\mathbf{r}')\psi_l(\mathbf{r})\langle \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l\rangle \end{aligned} \tag{3.3}$$

expressed in terms of single particle density matrix (SPDM) $\langle \hat{a}_i^\dagger \hat{a}_j\rangle$ and two particle density matrix (TPDM) $\langle \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l\rangle$. And any correlation functions can be calculated immediately from SPDM for first order, from TPDM for second order, from higher order density matrix elements for corresponding order where basis set $\{\psi_i(\mathbf{r})\}$ is given.

Though (3.3) itself is not useful way to calculate correlation functions when one need to tons of basis, e.g. momentum basis, however becomes quite plausible when there exist only few of basis to be considered in the system. A representative example of such system is Bose-Einstein Condensate (BEC), the many-body state of number N particles which $\mathcal{O}(N)$ particles occupy one mode (single particle state), call it here $\psi_0(\mathbf{r})$ thus $\langle \hat{a}_0^\dagger \hat{a}_0\rangle \sim \mathcal{O}(N)$, and else are not occupied by $\mathcal{O}(N)$ particles. Before we go further, since condensation happens and strengthens as $T \to 0$, we'd like to focus on ground state rather than finite temperature properties. Rewrite field operator expansion as follows

$$\hat{\psi}(\mathbf{r}) = \psi_0(\mathbf{r})\hat{a}_0 + \sum_{i=1}^{\infty} \psi_i(\mathbf{r})\hat{a}_i \tag{3.4}$$

For non interacting ideal gas, $i \neq 0$ part represents thermal depletion which has no coherence with $\psi_0(\mathbf{r})$ at all [12], thus for large N it is allowed to write as

$$\hat{\psi}(\mathbf{r}) = \sqrt{N}\psi_0(\mathbf{r}) + \sum_{i=1}^{\infty} \psi_i(\mathbf{r})\hat{a}_i \tag{3.5}$$

replacing $\hat{a}_0$ by $\sqrt{N}$. Comparing with $\hat{\psi}^\dagger(\mathbf{r})$, $\hat{a}_0^\dagger$ is also replaced by $\sqrt{N}$. An error occur by this approximation process comes from letting $[\hat{a}_0, \hat{a}_0^\dagger] = 1$ to be 0. So the error is typically $\mathcal{O}(1/N)$, and becomes almost negligible as N gets larger and even exact as $N \to \infty$ with interaction also [11]. For interacting gas, as can be seen from Bogoliubov approximation for weakly interacting Bose gas, $i \neq 0$ part also contains terms called as quantum depletion since ground state (which we might call as condensate here) of weakly interacting gas consists of not only one single particle state, but also other many states with each $\mathcal{O}(1)$ occupations and even destroying condensate when number of particles in quantum depletion part almost equals to or exceeds N [13].

Approximating field operator as in (3.5), $\hat{a}_0 \simeq \sqrt{N}$, $\hat{a}_0^\dagger \simeq \sqrt{N}$ not only just shows that there exists macroscopically (of $\mathcal{O}(N)$) occupied one mode, but also enables decomposition of SPDM. $\langle \hat{a}_0^\dagger \hat{a}_j \rangle = 0$ for $j \neq 0$, thus SPDM can be written as direct sum of $\langle \hat{a}_0^\dagger \hat{a}_0 \rangle = N_0$ and $\langle \hat{a}_i^\dagger \hat{a}_j \rangle$ of $i, j \neq 0$ where $\sum_{i \neq 0} \langle \hat{a}_i^\dagger \hat{a}_i \rangle = N - N_0$. Since occupation number for specific mode can be well defined in eigenbasis of SPDM ($\langle \hat{a}_i^\dagger \hat{a}_j \rangle = 0$ for $i \neq j$), this can be accepted as natural result when there actually is macroscopic mode.

Going further into TPDM $\langle \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l \rangle$, there exist three types of non zero TPDM elements.

1. $i, j, k, l = 0$
2. $i, j, k, l \neq 0$
3. $i = k = 0,\ j, l \neq 0$ up to commutation.

TPDM is not just direct sum of 'TPDM of macroscopic mode' and 'TPDM of non-macroscopic modes' as SPDM due 3. And case 3 also corresponds to a multiplication of two SPDM since $\langle \hat{a}_0^\dagger \hat{a}_j^\dagger \hat{a}_0 \hat{a}_l \rangle = N_0 \langle \hat{a}_j^\dagger \hat{a}_l \rangle$. Hence again whole TPDM of the system can be fully described by SPDM and TPDM of each macroscopic mode and non-macroscopic modes. Extra correlations between two different types of modes is not needed. Similar kind of logic can be stated for higher order density matrices, therefore allows to consider macroscopic mode (condensate) and non macroscopic mode (depletion) respectively. (3.5) contains more than just one macroscopic mode, and the existence of macroscopic mode alone is not sufficient condition to use (3.5) properly. As stated above, for SPDM macroscopic mode can be defined by occupation number, which has correct meaning when mode itself is constituent of eigenbasis of SPDM, anyway. For higher order density matrices we'd like to introduce the concept of coherence between modes to clarify sufficient condition to write (3.5).

Let us introduce following condition defined by arbitrary n-th order density matrix element.

$$\left\langle (\hat{a}_i^\dagger)^\alpha (\hat{a}_j^\dagger)^{n-\alpha} (\hat{a}_i)^\beta (\hat{a}_j)^{n-\beta} \right\rangle = 0 \text{ when } \alpha \neq \beta \tag{3.6}$$

State satisfying (3.6) is called as Fock state proportional to $(\hat{a}_i^\dagger)^{N_i} (\hat{a}_j^\dagger)^{N_j} |0\rangle$ except normalization factor. And it allows to write any correlation function consisting of i-th mode and j-th mode operators in terms of multiplication of two correlation function each depends on $\hat{a}_i, \hat{a}_i^\dagger$ and $\hat{a}_j, \hat{a}_j^\dagger$ as follows

$$\left\langle \hat{O}(\hat{a}_i, \hat{a}_i^\dagger, \hat{a}_j, \hat{a}_j^\dagger) \right\rangle = \left\langle \hat{O}_i(\hat{a}_i, \hat{a}_i^\dagger) \hat{O}_j(\hat{a}_j, \hat{a}_j^\dagger) \right\rangle = \left\langle \hat{O}_i(\hat{a}_i, \hat{a}_i^\dagger) \right\rangle \left\langle \hat{O}_j(\hat{a}_j, \hat{a}_j^\dagger) \right\rangle \tag{3.7}$$

A superposition state proportional to $(\sqrt{N_i}(\hat{a}_i^\dagger)^N + \sqrt{N_j}(\hat{a}_j^\dagger)^N) |0\rangle$ satisfies (3.6) except $n = N, \alpha = 0, \beta = N$ or $n = N, \alpha = N, \beta = 0$, however $\left\langle \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_i \hat{a}_j \right\rangle \neq N_i \left\langle \hat{a}_j^\dagger \hat{a}_l \right\rangle$ and it is 0 thus not able to write field operator as in (3.5). This time correlation function satisfies

$$\left\langle \hat{O}(\hat{a}_i, \hat{a}_i^\dagger, \hat{a}_j, \hat{a}_j^\dagger) \right\rangle = \left\langle \hat{O}_i(\hat{a}_i, \hat{a}_i^\dagger) \hat{O}_j(\hat{a}_j, \hat{a}_j^\dagger) \right\rangle = 0 \text{ unless } \hat{O}_i = 1 \text{ or } \hat{O}_j = 1 \tag{3.8}$$

This fact leads to

$$\left\langle \hat{O}(\mathbf{r}_1, \cdots, \mathbf{r}_n) \right\rangle = N_i \left\langle \hat{O}_i(\mathbf{r}_1, \cdots, \mathbf{r}_n) \right\rangle + N_j \left\langle \hat{O}_j(\mathbf{r}_1, \cdots, \mathbf{r}_n) \right\rangle \tag{3.9}$$

for any n-th order correlation function $\langle \hat{O}(\mathbf{r}_1, \cdots, \mathbf{r}_n) \rangle$ of the system which has only $N_i = \langle \hat{a}_i^\dagger \hat{a}_i \rangle$ and $N_j = \langle \hat{a}_j^\dagger \hat{a}_j \rangle$ as none 0 occupation number where two-modes i and j are eigenbasis of SPDM where

$$\begin{aligned} \hat{O}_i(\mathbf{r}_1, \cdots, \mathbf{r}_n) &= f_i(\psi_i^*, \psi_i; \mathbf{r}_1, \cdots, \mathbf{r}_n) \hat{O}_i(\hat{a}_i, \hat{a}_i^\dagger) \\ \hat{O}_j(\mathbf{r}_1, \cdots, \mathbf{r}_n) &= f_j(\psi_j^*, \psi_j; \mathbf{r}_1, \cdots, \mathbf{r}_n) \hat{O}_j(\hat{a}_j, \hat{a}_j^\dagger). \end{aligned} \tag{3.10}$$

And a sufficient condition to write (3.5) in addition to existence of macroscopic mode can be written as

$$\Big\langle (\hat{a}_0^\dagger)^\alpha \prod_{j=1}^{\infty} (\hat{a}_j^\dagger)^{\alpha_j} (\hat{a}_0)^\beta \prod_{k=1}^{\infty} (\hat{a}_k)^{\beta_k} \Big\rangle = 0 \text{ when } \alpha \neq \beta \tag{3.11}$$

where $\sum_j \alpha_j = n - \alpha, \ \sum_k \beta_k = n - \beta$ here. This enables us to consider correlation function for macroscopic mode independently from other non macroscopic modes, i.e. we can use truncated field operator $\hat{\psi}_c(\mathbf{r}) = \psi_0(\mathbf{r})\hat{a}_0 \simeq \sqrt{N_0}\psi_0(\mathbf{r})$ to describe the physics of condensation part independently. In Bogoliubov approximation, it is known that we can rather safely use (3.5). To sum up,

- Ground state of Bose gas can condense into one macroscopic mode even with interaction depending on condition.
- Correlation functions can be calculated from informations (=SPDM, TPDM, etc.) each about macroscopic mode and non macroscopic mode.
- It's possible to consider macroscopic mode independently through truncated operator $\hat{\psi}_c(\mathbf{r}) = \psi_0(\mathbf{r})\hat{a}_0 \simeq \sqrt{N_0}\psi_0(\mathbf{r})$ as long as macroscopic mode satisfies (3.11).

By considering (1.4) with truncated field operator $\hat{\psi}_c(\mathbf{r}) \simeq \sqrt{N_0}\psi_0(\mathbf{r})$ and contact interaction $V(\mathbf{r}, \mathbf{r}') = \frac{g}{2}\delta(\mathbf{r} - \mathbf{r}')$, one obtains Gross-Pitaevskii (GP) equation from functional derivate of $\psi_0^*(\mathbf{r})$ taken to one-mode Hamiltonian [3, 5].

$$\mu\psi_0(\mathbf{r}) = \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}) + gN_0|\psi_0(\mathbf{r})|^2 \right) \psi_0(\mathbf{r}) \tag{3.12}$$

where μ is called as chemical potential coming from normalization condition of $\psi_0(\mathbf{r})$. From this equation, one not only can determine optimized wavefunction ψ_0, but also can find allowed low-energy excitation structure as long as condensation is kept. And it shows unique nature of BEC such as vortex, existence of overall coherent phase as laser shown from interference experiment [14], long range spatial coherence [15].

Here we go one step further, by introducing more than one macroscopic mode. To deal with definition of many macroscopic modes, we need to start from SPDM especially in eigenbasis, we will call it as natural basis, now. If there exist more than one macroscopic eigenvalues of SPDM, we call such state as *fragmented state* [4, 10]. Then it is natural to think of truncating field operator $\hat{\psi}(\mathbf{r})$ for condensate part (part consist of macroscopically occupied modes) as

$$\hat{\psi}_c(\mathbf{r}) \simeq \sum_{i=0}^{M-1} \psi_i(\mathbf{r})\hat{a}_i \tag{3.13}$$

with condition extended from (3.11) as

$$\langle (\prod_{i=0}^{M-1} \hat{a}_i^\dagger)^{\alpha_i} \prod_{j=M}^{\infty} (\hat{a}_j^\dagger)^{\alpha_j} \prod_{k=0}^{M-1} (\hat{a}_k)^{\beta_k} \prod_{l=M}^{\infty} (\hat{a}_l)^{\beta_l} \rangle = 0 \text{ when } \sum_{i=0}^{M-1} \alpha_i \neq \sum_{k=0}^{M-1} \beta_k \tag{3.14}$$

especially when the number of macroscopic orbitals is M. As done for BEC, $M = 1$ case, we'd like to find lowest energy configuration with reduced Hamiltonian of $\hat{\psi}_c(\mathbf{r})$ for condensation part thus get optimized orbitals for modes, number of orbitals, occupation number configuration.

In previous $M = 1$ case, since GP equation has only one macroscopic mode, so finding best one orbital in functional space was enough and relatively not too hard where trap geometry determines density approximately (e.g. by Thomas-Fermi approximation) which is directly related to $|\psi_0(\mathbf{r})|^2$, and there isn't need for later two optimization. On the other hand, when $M > 1$ it is impossible to determine approximate magnitude of each orbitals from expected density solely. Degrees of freedom increases, and at the same time finding two optimized variational function in functional space is almost impossible itself without help of extra informations; as optical lattice well approximated by localized Wannier wavefunctions per each lattice sites.

Recently, there has been establishment of theories and numerical simulations regarding many macroscopic mode from the first principle [16, 17] and there has been some progresses including single-trap fragmentation with long range repulsive interaction [18]. However number of degree of freedom explodes as number of mode M gets larger and this fact still limits further utilization, e.g. for stronger interaction in a single trap case.

Instead, Here we consider $M = 2$ case with some assumption(s) on macroscopic orbitals following system's characteristic including trap geometry to avoid diverging difficulty of finding optimal orbitals in whole functional space. Now we use (3.13) to get following two-mode ($M = 2$) Hamiltonian, expansion from GP equation, which describes condensate part [8]

$$
\begin{aligned}
\hat{H} &= \sum_{i=0,1} \epsilon_i \hat{a}_i^\dagger \hat{a}_i - \left(\frac{\Omega}{2} \hat{a}_0^\dagger \hat{a}_1 + \text{h.c.} \right) + \frac{A_1}{2} \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 + \frac{A_2}{2} \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \\
&\quad + \left(\frac{A_3}{2} \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 + \text{h.c.} \right) + \frac{A_4}{2} \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 + \left(\frac{A_5}{2} \hat{a}_1^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 + \text{h.c.} \right) \\
&\quad + \left(\frac{A_6}{2} \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 + \text{h.c.} \right) \\
&= \sum_{i=0,1} \epsilon_i \hat{n}_i - \left(\frac{\Omega}{2} \hat{a}_0^\dagger \hat{a}_1 + \text{h.c.} \right) + \frac{A_1}{2} \hat{n}_0 (\hat{n}_0 - 1) + \frac{A_2}{2} \hat{n}_1 (\hat{n}_1 - 1) \\
&\quad + \frac{A_4}{2} \hat{n}_0 \hat{n}_1 + \left(\frac{A_3}{2} \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 + \text{h.c.} \right) + \left(\frac{A_5}{2} \hat{a}_1^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 + \text{h.c.} \right) \\
&\quad + \left(\frac{A_6}{2} \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 + \text{h.c.} \right)
\end{aligned}
\tag{3.15}
$$

with A_i defined as

$$
\begin{aligned}
& A_1 \equiv V_{0000},\ A_2 \equiv V_{1111},\ A_3 \equiv V_{0011},\ A_3^* \equiv V_{1100}, \\
& A_4 \equiv V_{0101} + V_{0110} + V_{1001} + V_{1010},\ A_5 \equiv V_{0001} + V_{0010}, \\
& A_5^* \equiv V_{1000} + V_{0100},\ A_6 \equiv V_{0111} + V_{1011},\ A_6^* \equiv V_{1110} + V_{1101} \\
& \text{where } V_{ijkl} \equiv \iint d\mathbf{r} d\mathbf{r}' \psi_i^*(\mathbf{r}) \psi_j^*(\mathbf{r}') V(\mathbf{r}, \mathbf{r}') \psi_k(\mathbf{r}') \psi_l(\mathbf{r})
\end{aligned}
\tag{3.16}
$$

and

$$
\epsilon_i = \int d\mathbf{r} \psi_i^*(\mathbf{r}) \hat{h} \psi_i(\mathbf{r}),\ \Omega = -2 \int d\mathbf{r} \psi_0^*(\mathbf{r}) \hat{h} \psi_1(\mathbf{r}) \tag{3.17}
$$

Once above coefficients are determined, it is not hard to get a state subject to certain condition (e.g. ground state). By applying following general two-mode ansatz to Hamiltonian,

$$|\Psi\rangle = \sum_{l=0}^{N} C_l \frac{(\hat{a}_0^\dagger)^{N-l}(\hat{a}_1^\dagger)^l}{\sqrt{(N-l)!l!}} |0\rangle \equiv \sum_{l=0}^{N} C_l \, |N-l, l\rangle \,. \tag{3.18}$$

C_l, which is equivalent to complete information of two-mode state except orbitals, can be calculated from rather easy numerics in many case. The problem is that coefficients depend on the choice of orbitals $\psi_0(\mathbf{r}), \psi_1(\mathbf{r})$ of which optimal choice is almost impossible to be revealed without extra conditions. Thus here we will consider rather predetermined orbitals from Wannier function of double well case, or introducing common one variational scalar parameter, which is size of orbitals, to reduce divergent degree of freedom considered for full optimization of orbitals in functional space.

3.2 Fragmented State in a Single Trap with Even-Odd Parity

Now we move on to our main subject, single-trap fragmented state in (quasi) 1D geometry. In the case of single-trap geometry, it is hard to show whether fragmented state can be ground state instead of BEC for weak interaction strength or Fermionization for extremely strong interaction in 1D. Claims for existence owe will consider here single-trap fragmented state of negative coherence, which were done in existing studies [8, 19, 20, 21]. Here we will introduce the idea briefly.

Thinking of harmonic trap without interaction, ground state is determined by balancing kinetic energy E_{K} and potential energy E_{P}. After repulsive interaction is turned on, spatial width (extension) of ground state will increase with interaction. This weakens E_{K} where gradient of wavefunction decreases, and new ground state is determined by balancing E_{P} and E_{I} which is called as Thomas Fermi (TF) limit. In the case of homogeneous square-well trap, hand waving argument on why only one mode description is favored as ground state in this translational invariant trap geometry is explained in [24] up to Hartree-Fock limit with contact interaction. However, inhomogeneous trap allows lowering energy with more than one macroscopic mode thus still there remains possibility for single-trap fragmented state. Considering that a mode orthogonal to first mode of even function with minimal width (extension) to decrease E_{P} should be odd function, assuming two orbitals to have even (ψ_0) and odd (ψ_1) parity with significant magnitude overlap is reasonable trial for single-trap geometry without optimization of orbitals with heavy or impossible calculation. Also, this model allows smooth connection from BEC (Only even function is macroscopic) to fragmented state (both even and odd functions are macroscopic). Even and odd orbitals choice gives $\Omega = A_5 = A_6 = 0$. Two-mode Hamiltonian is

$$
\begin{aligned}
\hat{H} =& \epsilon_0 \hat{a}_0^\dagger \hat{a}_0 + \epsilon_1 \hat{a}_1^\dagger \hat{a}_1 + \frac{A_1}{2} \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 + \frac{A_2}{2} \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \\
&+ \left(\frac{A_3}{2} \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 + \text{h.c.} \right) + \frac{A_4}{2} \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0^\dagger \hat{a}_0
\end{aligned}
\tag{3.19}
$$

Plugging in $\langle \Psi |$ and $| \Psi \rangle$ of (3.18) into above equation and taking derivative with respect to C_l^* gives following energy equation which C_l of ground state satisfies

$$
\begin{aligned}
EC_l =& \frac{1}{2} A_3 (d_l C_{l+2} + d_{l-2} C_{l-2}) + C_l \big[\epsilon_0 (N-l) \\
&+ \epsilon_1 l + \frac{1}{2} A_1 (N-l)(N-l-1) + \frac{1}{2} A_2 l(l-1) + \frac{1}{2} A_4 (N-l) l \big]
\end{aligned}
\tag{3.20}
$$

where $d_l = \sqrt{(l+2)(l+1)(N-l-1)(N-l)}$ (which seems to be missed in recent PRL). Taking continuum limit from C_l to $C(l)$ in (3.20), one gets $|C(l)|$ distribution as

$$
|C(l)| = \left(\pi a^2 \right)^{-1/4} \exp[-\left(l - N/2 - \mathcal{S} \right)^2 / (2 a_{\mathrm{osc}}^2)]. \tag{3.21}
$$

where Gaussian distribution width a_{osc} and shift $\mathcal{S}$ is given by two-mode Hamiltonian coefficients [8, 19] as,

$$
a_{\mathrm{osc}}^2 = N \sqrt{\frac{|A_3|}{A_1 + A_2 + 2A_3 - A_4}}, \quad \mathcal{S} = \frac{N(A_1 - A_2)/2 + \epsilon_0 - \epsilon_1}{A_1 + A_2 + 2A_3 - A_4} \tag{3.22}
$$

Here we assumed $C_l \in \mathbb{R}$ and further extension for complex C_l can be found in the following and in [21]. C_l and $C_{l\pm 2}$ will have no sign change and thus become BEC when $A_3 < 0$, and will have negative sign thus fragmented when $A_3 > 0$.

Also be aware of (3.20) doesn't give any constraint between C_l and $C_{l\pm1}$ which leads to decoupled C_l of even l (we will call as an even sector) and C_l of odd l (odd sector). For $A_3 > 0$, we get TPDM elements to $\mathcal{O}(1/N)$ ($N_i = \langle \hat{a}_i^\dagger \hat{a}_i \rangle$)

$$\begin{aligned} &\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 \rangle = N_0^2,\ \langle \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \rangle = N_1^2, \\ &\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 \rangle = N_0 N_1,\ \langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \rangle = -N_0 N_1. \end{aligned} \tag{3.23}$$

Above relation is valid if $N_0^2, N_1^2 \gg N$, and more detailed discussion on TPDM element will be in chapter 5 and chapter 7. Before considering related correlation functions, let us discuss quasi-1d case. In quasi-1d, x and y direction is strongly suppressed (frozen) thus there only exists one single particle mode per each direction; call $\psi_{x0}(x)$ and $\psi_{y0}(y)$. we can consider field operator effectively as

$$\hat{\psi}(\mathbf{r}) = \psi_{x0}(x)\psi_{y0}(y)\left(\psi_0(z)\hat{a}_0 + \psi_1(z)\hat{a}_1 + \cdots\right) \tag{3.24}$$

where $\cdots$ can contain quantum fluctuation part and thermal gas part will be ignored here unless frozen direction is 'melted' with strong perturbation. Now truncation up to two modes yields

$$\hat{\psi}(\mathbf{r}) \simeq \psi_{x0}(x)\psi_{y0}(y)\left(\psi_0(z)\hat{a}_0 + \psi_1(z)\hat{a}_1\right) \tag{3.25}$$

thus it is enough to consider $\hat{\psi}(z)$ instead of $\hat{\psi}(\mathbf{r})$ in calculating correlation function, while x and y direction can be integrated out.

It is important to note following caution with truncation of field operator. To apply truncated field operator in calculating correlation function, corresponding operator should be normal ordered. For detailed discussion, see appendix A. Now we write two correlation function in two-mode expression. Density $\rho_1(z)$

$$\rho_1(z) = \langle \hat{\psi}^\dagger(z)\hat{\psi}(z) \rangle = N_0|\psi_0(z)|^2 + N_1|\psi_1(z)|^2 \tag{3.26}$$

and density-density correlation function $\rho_2(z,z')$ with $\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$

$$\begin{aligned}
&\rho_2(z,z') = \langle \hat{\psi}^\dagger(z)\hat{\psi}^\dagger(z')\hat{\psi}(z')\hat{\psi}(z)\rangle \\
&=|\psi_0(z)|^2|\psi_0(z')|^2\langle \hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_0\hat{a}_0\rangle + 0 \to 1 + \left(|\psi_0(z)|^2|\psi_1(z')|^2 + 0 \leftrightarrow 1\right)\langle \hat{n}_0\hat{n}_1\rangle \\
&+2\Re\left[\psi_0^*(z)\psi_1^*(z')\psi_0(z')\psi_1(z)\langle \hat{n}_0\hat{n}_1\rangle + \psi_0^*(z)\psi_0^*(z')\psi_1(z')\psi_1(z)\langle \hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_1\hat{a}_1\rangle\right].
\end{aligned} \tag{3.27}$$

3.2.1 Negative Pair Coherent (NPC) State

From energy equation, for $A_3 > 0$, $\mathrm{Sgn}(C_l C_{l+2}) = -1$ is obtained together with (3.22). But (3.22) is the case when C_l is continuous and $C_l \in \mathbb{R}$. As stated already, absence of relation between C_l and C_{l+1} disconnects even C_l sector and odd C_l sector. Thus there is extra degree of freedom to change relative weight and relative phase between the two sectors. This leads to almost degenerated two fragmented states $|\mathrm{Even}\rangle$ and $|\mathrm{Odd}\rangle$

$$|\mathrm{Even}\rangle = \sum_{l=0,2,\cdots} C_{even,l}\,|N-l,l\rangle\,,\ \ |\mathrm{Odd}\rangle = \sum_{l=1,3,\cdots} C_{odd,l}\,|N-l,l\rangle \tag{3.28}$$

with $|C_{even,l}|, |C_{odd,l}|$ being $\sqrt{2}$ times of $|C_l|$ given in (3.22) satisfying normalization. Both states have $\langle \hat{a}_0^\dagger\hat{a}_1\rangle = 0$ thus they are fragmented states. Where interaction energy is about the same, energy difference between two states is given by $\epsilon_1 - \epsilon_0$ where $|\mathrm{Odd}\rangle$ has one more particle at $|\mathrm{Even}\rangle$. $\epsilon_1 - \epsilon_0$ is tiny amount $\mathcal{O}(1/N)$ or less comparing to total energy of system, therefore two states can be considered as degenerated

This effective degeneracy brings has more general class of ground state for the system considered in 3.2 as general superposition of two fragmented states

$$|\mathrm{NPC}\rangle = c(|\mathrm{Even}\rangle + ue^{i\theta_k}\,|\mathrm{Odd}\rangle \tag{3.29}$$

which we will call as negative pair coherent (NPC) state with normalization condition $|c|^2(1+|u|^2) = 1$ and $c, u \in \mathbb{R}, u \geq 0$. Here θ_k stands as relative phase between $|\mathrm{Even}\rangle$ and $|\mathrm{Odd}\rangle$. We'd like to note that for simplicity we take continuum limit, which does not hurt following related discussions and arguments later. More general definition of $|\mathrm{Even}\rangle$, $|\mathrm{Odd}\rangle$ and discussion for NPC state is in [21] which is not out of scope with continuum limit here.

$|\mathrm{NPC}\rangle$ has interesting property; it is not fragmented state for certain value of u and θ_k even though itself is superposition of two fragmented states. Let us calculate first order coherence $\left\langle \hat{a}_0^\dagger \hat{a}_1 \right\rangle = \langle \mathrm{NPC}| \hat{a}_0^\dagger \hat{a}_1 |\mathrm{NPC}\rangle$.

$$\begin{aligned} \left\langle \hat{a}_0^\dagger \hat{a}_1 \right\rangle =& |c|^2 \left(u e^{i\theta_k} \langle \mathrm{Even}| \hat{a}_0^\dagger \hat{a}_1 |\mathrm{Odd}\rangle + u^* e^{-i\theta_k} \langle \mathrm{Odd}| \hat{a}_0^\dagger \hat{a}_1 |\mathrm{Even}\rangle \right) \\ \simeq& 2N_0 N_1 i |c|^2 u \sin\theta_k \end{aligned} \tag{3.30}$$

therefore there exists non vanishing imaginary first order coherence, leading to decrease of degree of fragmentation $\mathcal{F}$ defined in (2.1). For $\theta_k = \pi/2, 3\pi/2$ with $u = 1$ we even have $\mathcal{F} = 0$. However, it should not be mislead as BEC since there exists non trivial negative pair coherence. This peculiar state appears in dynamical sweeping of A_3 which is crucial element in fragmentation in a single trap, from non fragmented region $A_3 < 0$ to $A_3 > 0$ [21].

3.2.2 Stability of Fragmented State

$|\mathrm{NPC}\rangle$ is obtained as ground state of the system with perfect even-odd parity without single particle tunneling $\hat{a}_0^\dagger \hat{a}_1$. Therefore it is worthwhile to summarize existing research about stability of fragmented state done in [20]. In [20], various type of perturbations were brought to test robustness of fragmented state. These are

- Fluctuation of occupation number of modes and fluctuation of relative phase between modes
- Josephson-type tunneling $\hat{a}_0^\dagger \hat{a}_1$
- Breaking even-odd parity with deformation of additional mode 1
- Additional third mode
- Finite temperature effect

and it was shown that symmetric or anti-symmetric combination $|\mathrm{Even}\rangle \pm |\mathrm{Odd}\rangle$ are stable against those perturbation unless strength of perturbation is too strong. For detail, refer to [20]. And these combination states, one of these is ground state depending on perturbation parameter, are fragmented state of maximal fragmentation since $\left\langle \hat{a}_0^\dagger \hat{a}_1 \right\rangle = 0$. And one of $|\mathrm{Even}\rangle \pm |\mathrm{Odd}\rangle$ is many-body ground state, therefore it is legitimate to consider one of these state to be actual many-body state of system in static case (For dynamic case it is not true [21]). Thus these two state of *continuous* $C_l \in \mathbb{R}$ will be the state of interest in the next chapter.

It is noted that, further investigation on stability of fragmented state can be done with following depletion and decoherence sources

1. Quantum depletion $\delta\hat{\psi}(z)$ with $\hat{\psi}(z) = \psi(z)\hat{a}_0 + \psi_1(z)\hat{a}_1 + \delta\hat{\psi}(z)$.

2. Three-body recombination.

Quantum depletion can possibly be investigated with linear response theory starting from [16] or [17], which is self-consistent fully variational many-mode theory. There is already research on linear response of double well fragmented state [25]. However difficulty rises ridiculously when two orbitals, spatial wave-function of mode, have large overlap in magnitude to each other in a single trap and mean-field does not work at all.

Since interaction strength plays crucial role for emergence of fragmentation, fragmented state in a single trap will be much more realistic option if one can show that the fragmented state is stable against quantum depletion. And in the case of three-body recombination, which is related to lifetime of condensate, recombination rate increases with interaction strength thus it is reasonable research to investigate stability of fragmented state.

Chapter 4

Detecting the Single-Trap Fragmentation

In the previous chapter, a possible existence of the fragmentation in a single trap was shown by arguing that increasing interaction strength allows a state with finite degree of fragmentation $\mathcal{F}$ defined in (2.1) to have lower energy than $\mathcal{F} = 0$ state which occupies ψ_0 mode only, despite of finite single particle energy gap $\epsilon_1 - \epsilon_0 > 0$ between ψ_0 and ψ_1.

As discussed previously, under the existence of interaction it is hard to prove the (single-trap) fragmented state as an exact ground state of the given system by solving fully self-consistent quantum many-body equation. Therefore, it is worthwhile and feasible to find distinguishing feature, in certain correlation function, of the fragmented state which can be measured in an experiment. In ultracold quantum gas experiment, systems are usually considered to be in thermal gas state no macroscopic mode, or BEC with one macroscopic mode with increasing fraction of condensation as temperature T decreases. Detecting such property of a many-body state in an experimental measurement will discriminate the many-body stated from BEC or thermal gas, thus prove the existence of the fragmented state. We start from following properties of the single-trap fragmented state in quasi-1d which was claimed in previous chapter

1. ψ_0 and ψ_1 has a large overlap quantified by $\int d\mathbf{z}\, |\psi_0(\mathbf{z}')||\psi_1(\mathbf{r})|$
2. Even (ψ_0) - odd (ψ_1) parity
3. $\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \rangle \simeq -\langle \hat{n}_0 \hat{n}_1 \rangle$ and $\langle \hat{a}_0^\dagger \hat{a}_1 \rangle$, $\langle \hat{n}_0 \hat{a}_0^\dagger \hat{a}_1 \rangle$, $\langle \hat{n}_1 \hat{a}_0^\dagger \hat{a}_1 \rangle \simeq 0$.

In this chapter, we consider the single-trap fragmented state to satisfy above three properties. It is anticipated to find distinguishable feature of the fragmented state within measurable correlation function from above imposed conditions and to reveal how such feature of correlation function will appear in real experiment.

Spatial coherence $\rho_1(z,z') = \langle \hat{\psi}^\dagger(z)\hat{\psi}(z')\rangle$ and density-density correlation $\rho_2(z,z') = \langle \hat{\psi}^\dagger(z)\hat{\psi}^\dagger(z')\hat{\psi}(z')\hat{\psi}(z)\rangle$ are treated as possible candidates showing unique feature of the fragmented state distinguished from BEC and thermal gas.

For fragmented state in a single trap, density-density correlation $\rho_2(z,z')$, where above imposed three properties show that $\rho_2(z,z')$ decreases drastically along $z=-z'$ and increases drastically along $z=z'$ after Time-of-Flight (TOF). This tendency is directly related to the degree of fragmentation $\mathcal{F}$, and $\rho_2(z,-z)$ reaches almost 0 for maximal fragmentation $\mathcal{F}=1$ which cannot be observed with BEC. For quasi-1d case, $\rho_2(z,z') \to \rho_2(z,z')$ is visualized on the $z-z'$ plane, with simple harmonic oscillator ground state (ψ_0, Gaussian) and 1st excited state (ψ_1). At the end of this chapter, $\rho_2(z,z')$ for the single-trap fragmented state is dealt with different ψ_0, ψ_1 to examine whether small changes in ψ_0, ψ_1 affect the result with simple harmonic oscillator eigenstates; suppression of $\rho_2(z,z')$ along $z=-z'$ and enhancement of $\rho_2(z,z')$ along $z=z'$.

4.1 Spatial Coherence

Spatial coherence can determine since thermal gas obeys Bose-Einstein statistics over T_c has correlation length, defined from exponential (Gaussian) decay of $\rho_1(z,z')$, for weakly interacting case which is a scale of thermal De Broglie wavelength λ_T. This leads to absence of spatial coherence for $|z-\mathbf{r}'| \gg \lambda_T$ thus non vanishing spatial coherence over $|z-\mathbf{r}'| \gg \lambda_T$ indicate that the system has certain condensation phenomena which means that there exists BEC or fragmented state. Also, spatial coherence is effective to detect a fragmented state with spatially separated ψ_0 and ψ_1, e.g. fragmented state in a double well with high barrier. However it does not work for the single-trap fragmented state case due to large overlap between ψ_0 and ψ_1. Looking at Fig.4.1 (here we consider quasi-1d case),

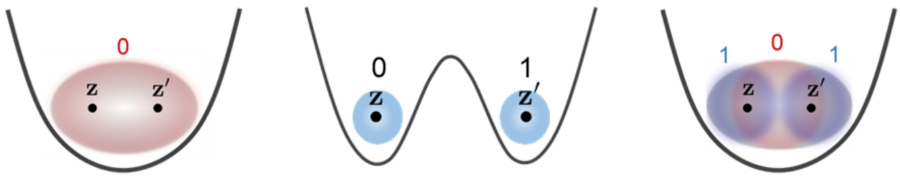

Figure 4.1: Schematic figure on measurement of $\rho_1(z,z')$ for BEC (left), double well fragmented state (center), fragmented state in a single trap (right).

1. BEC: $\rho(z), \rho(z') \neq 0 \;\rightarrow\; \rho_1(z,z') \neq 0$
   $\rho_1(z,z') = N_0\psi_0^*(z)\psi_0(z') \sim N_0\sqrt{\rho(z)\rho(z')}$
2. Double well fragmented state: $\rho(z), \rho(z') \neq 0$ but $\rho_1(z,z') = 0$
   $\rho_1(z,z') = N_0\psi_0^*(z)\psi_0(z') + N_1\psi_1^*(z)\psi_1(z)$
3. Fragmented state in a single trap: $\rho(z), \rho(z') \neq 0 \;\rightarrow\; \rho_1(z,z') \neq 0$
   $\rho_1(z,z') = N_0\psi_0^*(z)\psi_0(z') + N_1\psi_1^*(z)\psi_1(z)$

Purpose of measurement concerned here is to discriminate fragmented state in a single trap from BEC, therefore spatial coherence is inadequate.

4.2 Density-Density Correlation in the Single-Trap Fragmentation

Now it is time to examine density-density correlation $\rho_2(z,z') = \langle\hat{\rho}(z)\hat{\rho}(z')\rangle \simeq \langle\hat{\psi}^\dagger(z)\hat{\psi}^\dagger(z')\hat{\psi}(z')\hat{\psi}(z)\rangle$ TPDM elements from the continuum limit for C_l up to $\mathcal{O}(1/N)$ are

$$\langle\hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_0\hat{a}_0\rangle = N_0^2,\ \langle\hat{a}_1^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_1\rangle = N_1^2,\ \langle\hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0\rangle = N_0N_1,\ \langle\hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_1\hat{a}_1\rangle = -N_0N_1. \tag{4.1}$$

This result remains valid as long as the C_l distribution is centered at $l_0 \sim \mathcal{O}(N)$ with a width $\ll N$ which is the case for the single-trap fragmented state considered now. Then $\rho_2(z,z')$ equals to

$$\langle\hat{\rho}(z)\rangle\langle\hat{\rho}(z')\rangle + 2\psi_0(z)\psi_0^*(z')\psi_1^*(z')\psi_1(z)\left(\langle\hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0\rangle + \langle\hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_1\hat{a}_1\rangle\right) \tag{4.2}$$

and second term vanishes from $\langle\hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0\rangle = -\langle\hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_1\hat{a}_1\rangle$, yielding

$$\rho_2(z,z') = \langle\hat{\rho}(z)\rangle\langle\hat{\rho}(z')\rangle. \tag{4.3}$$

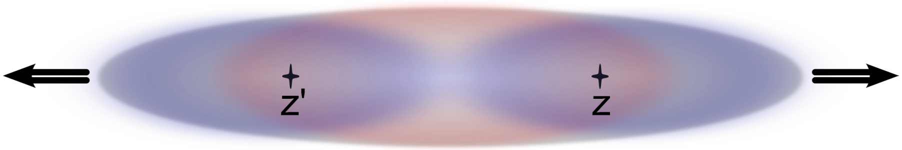

Figure 4.2: Schematic figure on TOF of fragmented state in quasi-1d geometry [22].

Due to large overlap, it is hard to discriminate fragmented state from BEC. Further, vast amount of difficulties in full calculation tells us that it would be impossible to calculate detailed shape of orbitals. Thus density-density correlation $\rho_2(z,z')$ is also inadequate quantity due to cancellation between two TPDM element including pair coherence $\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \rangle$.

However, Time-of-Flight (TOF) changes whole landscape. Turning off the trap potential in the weakly confining axial direction only [23], cf. Fig. 4.2, after a short initial period of rapid expansion, for $t \gg 1$, the gas will expand ballistically (See appendix B for detail)). One can then apply the noninteracting propagator to the initial orbitals

$$\psi_j(z,t) \quad = \quad \sqrt{\frac{1}{2\pi i w_t^2}} \exp\left[\frac{iz^2}{2w_t^2}\right] \tilde{\psi}_j(z,t); \quad w_t = \sqrt{t}. \tag{4.4}$$

where $\tilde{\psi}_j(z,t) = \exp[\frac{-iz^2}{2w_t^2}] \int dz' \psi_j(z',0) \exp[\frac{i(z-z')^2}{2w_t^2}]$. At late times, $\tilde{\psi}_j(z,t)$ has the meaning of a Fourier transform with respect to the variable pair $(z', z/w_t^2)$ to first order of z'/w_t, $\psi_j(z',0)$ remaining spatially confined.

Impressive point is that this TOF evolution introduces $\pi/2$ rotation of relative phase between $\tilde{\psi}_0$ and $\tilde{\psi}_1$. Now $\rho_2(z,z',-t)$ equals to

$$\langle \hat{\rho}(z,t) \rangle \langle \hat{\rho}(z',t) \rangle + 2\tilde{\psi}_0(z,t)\tilde{\psi}_0^*(z',t)\tilde{\psi}_1^*(z',t)\tilde{\psi}_1(z,t) \left(\langle \hat{n}_0 \hat{n}_1 \rangle - \langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \rangle \right) \tag{4.5}$$

and this time $\langle \hat{n}_0 \hat{n}_1 \rangle = \langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 \rangle = -\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \rangle$ *enhances* second term twice. Due to odd parity of $\tilde{\psi}_1$, correlation along $z = z'$ and $z = -z'$ are under very opposite effect of second term in (4.5). Correlation ratio $\mathcal{R}(z,t)$ among

$z, z' = -z$ for $t \gg 1$ is

$$\mathcal{R}(z,t) \equiv \frac{\rho_2(z,-z,t)}{\rho_2(z,z,t)} = \frac{(|\tilde{\psi}_0(z,t)|^2 N_0 - |\tilde{\psi}_1(z,t)|^2 N_1)^2}{\rho^2(z,t) + 4N_0N_1|\tilde{\psi}_0(z,t)|^2|\tilde{\psi}_1(z,t)|^2}. \tag{4.6}$$

According to the above formula, the approximately vanishing value of $\mathcal{R}(z,t)$ for large $\mathcal{F}$, visible both in Fig. 4.3 and Fig. 4.4, is related to comparable initial curvature radii of modes with given parity, i.e., to comparable dominant Fourier components. Note that $\rho_2(z,-z,t)/\rho_2(z,z,t) = 1 \;\forall\, t$ when there is no fragmentation ($N_0 = N$).

We stress that when the pair coherence $\left\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \right\rangle$ + h.c. [last term in Eq. (4.1)] were set positive, the ratio in (4.21) becomes unity. The corresponding large difference in the ratio of off-diagonal to diagonal density-density correlations thus allows for the confirmation of the negative sign of the macroscopic pair-coherence $\propto \mathcal{O}(N^2)$.

We now make our discussion explicit by assuming the following initial orbitals set. The harmonic oscillator ground state is used for the lower single-particle state, $\psi_0(z) = \pi^{-1/4} \exp\left[-z^2/2\right]$. For the excited (odd) state, we construct a superposition of two Gaussians of opposite sign and the same width, with symmetrically placed centers a distance d apart. This leads to

$$\psi_1(z) = \frac{1}{\pi^{1/4}} \frac{\sinh\left(zd/2\right) \exp\left[-z^2/2\right]}{\exp\left[d^2/16\right] \sqrt{\sinh\left(d^2/8\right)}}. \tag{4.7}$$

Varying d, this choice serves to illustrate the influence of the overlap of the moduli $|\psi_{0,1}(z)|$ on the correlations. For $d \to 0$ we obtain simply the first excited harmonic oscillator state, $\psi_1(z) \to \pi^{-1/4}\sqrt{2} z \exp\left[-z^2/2\right]$, for $d \gg 1$ the outer peaks are located where the central Gaussian $\psi_0(z)$ has essentially zero weight, cf. left sides of Fig. 4.3 and Fig. 4.4.

Two major factors related to strong fluctuation were negative pair coherence $\left\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \right\rangle$ and $\pi/2$ rotation of relative phase between two-mods ψ_0 and ψ_1 during TOF. Combination of two factors generated dramatic change after TOF, suppressing $\mathcal{R}(z,t)$.

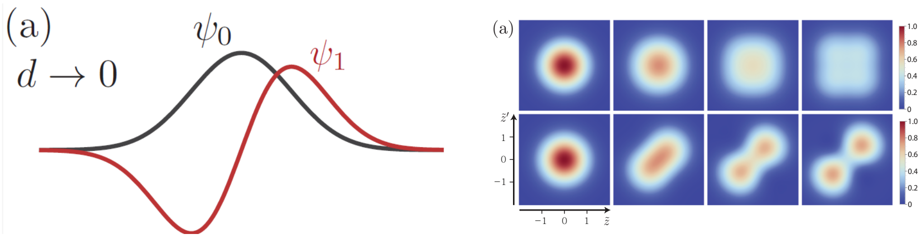

Figure 4.3: Two orbitals $\psi_0(z)$ and $\psi_1(z)$ for $d = 0$ (Left), and corresponding $\rho_2(z, z')$ for various values of $\mathcal{F}$ increasing from left to right (Right) [22].

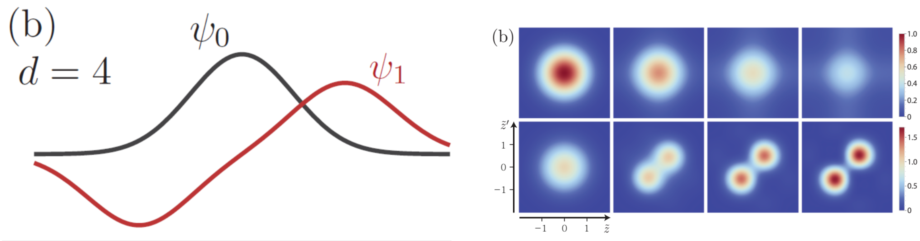

Figure 4.4: Two orbitals $\psi_0(z)$ and $\psi_1(z)$ for $d = 4$ (Left), and corresponding $\rho_2(z, z')$ for various values of $\mathcal{F}$ increasing from left to right (Right) [22].

4.3 Different Orbitals Case

The result above was derived for specific orbitals set; ground and 1st excited state of simple harmonic oscillator. So there is a question whether this result can be applied for different choice of orbitals satisfying even-odd parity. The reason why we get neat and simple expression for density-density correlation as (4.6) is that each $\psi_i(z,t)$ having position dependent phase becomes $\psi_{it}(z)$ which has constant phase along z (up to sign change at $z = 0$ in the case of odd function) and invariant shape up to rescale. This originates from the fact that phase factor $\phi(z,t)$ cancels each other and $\delta(t) \to 0$ for $t \gg \omega^{-1}$. To apply this argument to other shape of orbitals, first let's assume that orbitals are real for all z at $t = 0$ (up to constant phase). Looking at (B.13), one sees that in the long time limit any even and odd states gets constant phase difference irrelevant of z. (*Here I let the long time limit to be* $w_t \equiv \sqrt{\hbar t/m} \gg w$ *where* w *is width scale of wavefunction*)

Letting $\psi_0(z, t = 0)$ Thomas Fermi (TF) function $\sqrt{\frac{3(w^2-z^2)}{4w^3}}$ $(-w < z < w)$ of width w, TOF evolution cannot be calculated as analytic expression of w and

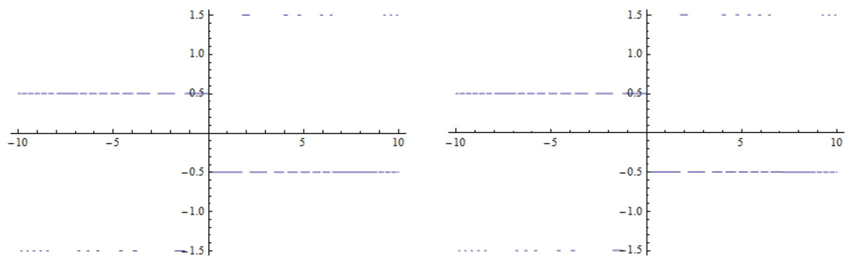

Figure 4.5: Argument of $\psi_0^*(z,t)\psi_1(z,t)$ divided by π for when $\psi_0(z,t=0)$ is ground state of simple harmonic oscillator and $\psi_0(z,t=0)=\frac{3(w^2-z^2)}{4w^3}$ (right)

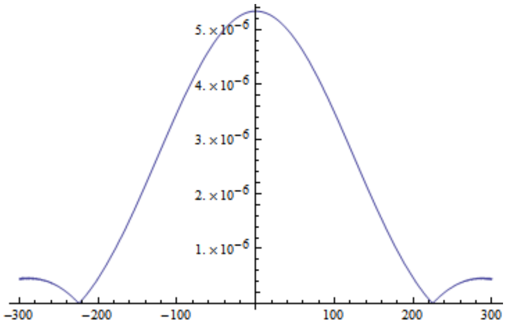

Figure 4.6: Absolute value of $\psi_0(z,t)$ when $\psi_0(z,t=0)=\frac{3(w^2-z^2)}{4w^3}$ (right)

z. Instead first applying for $\psi_0(z,t=0)=\frac{3(w^2-z^2)}{4w^3}$ and keeping $\psi_1(z,t=0)$ to be first excited state of simple harmonic oscillator, argument of $\psi_0^*(z,t)\psi_1(z,t)$ is shown in Fig.4.5.

Considering that 2π difference gives same argument, it is seen that argument for $z<0$ is $\pi/2$ and for $z>0$ is $-\pi/2$ (π difference happens for odd parity of $\psi_0^*(z,t)\psi_1(z,t)$, sign flip) for both case. And absolute value of $\psi_0(z,t)$ when $\psi_0(z,t=0)=\frac{3(w^2-z^2)}{4w^3}$ is shown in Fig.4.6.

Thus it is expected to observe significant decrease of $\mathcal{R}(z,t)$ for long time t also for $\psi_0(z,t=0)=\frac{3(w^2-z^2)}{4w^3}$ instead of ground state of simple harmonic oscillator. And also for TF function (and for other functions, too), we still can get approximate analytic expression in the limit of $w_t \gg w$. Propagated wavefunction $\psi(z,t)$ in terms of initial wavefunction $\psi(z',t=0)$ and w_t of free propagator ($g = 0$) is

$$\psi(z,t) = \sqrt{\frac{1}{2\pi i w_t^2}} \int dz'\, \psi(z',0)\, \mathrm{Exp}\left[\frac{i(z-z')^2}{2w_t^2}\right] \tag{4.8}$$

Let $\psi(z', t = 0)$ to be a TF function $\sqrt{\frac{3(w^2-z'^2)}{4w^3}}$. By writing exponential as follows,

$$\mathrm{Exp}\left[\frac{i}{2}\left(\frac{z}{w_t} - \frac{z'}{w_t}\right)^2\right] = \mathrm{Exp}\left[\frac{i}{2}\frac{z^2}{w_t^2}\right] \mathrm{Exp}\left[-i\frac{zz'}{w_t^2}\right] \mathrm{Exp}\left[\frac{i}{2}\frac{z'^2}{w_t^2}\right] \tag{4.9}$$

Expanding last exponential around $z' = 0$ we get

$$\mathrm{Exp}\left[\frac{i}{2}\frac{z^2}{w_t^2}\right] \mathrm{Exp}\left[-i\frac{zz'}{w_t^2}\right] \left(1 + \frac{i}{2}\frac{z'^2}{w_t^2} - \frac{1}{8}\frac{z'^4}{w_t^4} + \cdots\right) \tag{4.10}$$

As t gets larger and $w_t \gg w$, z'/w_t becomes very small since $-w < z' < w$ thus one or two terms in bracket of (4.10) give good approximation. Looking at following integration,

$$\begin{aligned} &\int_{-w}^{w} \sqrt{\frac{3\,(w^2 - z^2)}{4w^3}} \left(1 + \frac{iz'^2}{2w_t^2}\right) \mathrm{Exp}\left[-i\frac{zz'}{w_t^2}\right] dz \\ &= \frac{\sqrt{3}\pi \left(\left(2w_t^2 + iw^2\right) z J_1\left(\frac{wz}{w_t^2}\right) - 3iww_t^2 J_2\left(\frac{wz}{w_t^2}\right)\right)}{4\sqrt{w} z^2} \end{aligned} \tag{4.11}$$

it is seen that with Bessel function $J_1\left(\frac{wz}{w_t^2}\right)$ and $J_2\left(\frac{wz}{w_t^2}\right)$ one can express time evolution of TF function in TOF. (*For higher orders, it seems that J_3 and other Bessel function with higher moment does not appear, instead coefficient in front of Bessel function J_1 and J_2 changes with higher orders in z*) Actually, if w_t is 10 times of w, taking only 0th order of Taylor expansion is enough which is clear from below figure.

Taking 0th order approximation for the bracket in (4.10), $\psi_0(z,t)$ is,

$$\psi_0(z,t) \simeq \sqrt{\frac{1}{2\pi i w_t^2}} \frac{\sqrt{3}\pi w_t^2 J_1\left(\frac{wz}{w_2^2}\right)}{2\sqrt{w}z} \mathrm{Exp}\left[\frac{i}{2}\frac{z^2}{w_t^2}\right] \tag{4.12}$$

Looking at Fig.4.7, once again we can expect that $\mathcal{R}(z,t)$ will significantly decrease if argument between $\psi_0(z,t)$ and $\psi_1(z,t)$ does not depend on z for

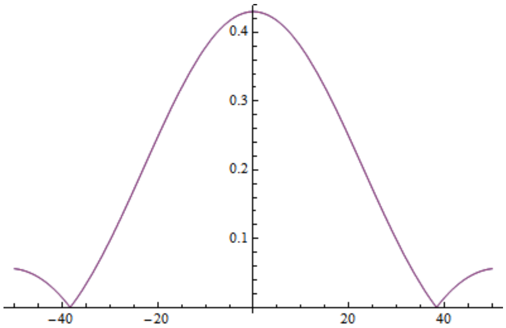

Figure 4.7: Two absolute values of $\psi_0(z,t)$ when $\psi_0(z,t=0)$ is TF function for $w_t = 10w$, one is up to 0th order and another is up to 1st order in (4.10). It is seen that they are of the 'same'.

large t. Let $\psi_1(z,t=0)$ to be 1st excited state of simple harmonic oscillator, recalling (B.6), as t gets larger phase factor $\phi(z,t)$ converges to $\frac{z^2}{w_t^2}$ which coincides with exponent in (4.12) and J_1 is real function which confirms that $\text{Arg}[\psi_0(z,t),\psi_1(z,t)]$ will not change with z for large t.

Also now it is possible to present other version of (B.14). Expanding $\text{Exp}\left[-i\frac{zz'}{w_t^2}\right]$ in (4.9) around $z'=0$, we get following expression for large t.

$$\begin{aligned} &\text{Exp}\left[\frac{i}{2}\frac{z^2}{w_t^2}\right]\left(1 - i\frac{zz'}{w_t^2} - \frac{1}{2}\left(\frac{zz'}{w_t^2}\right)^2 + \cdots\right)\text{Exp}\left[\frac{i}{2}\frac{z'^2}{w_t^2}\right] \\ \simeq\ &\text{Exp}\left[\frac{i}{2}\frac{z^2}{w_t^2}\right]\left(1 - i\frac{zz'}{w_t^2} - \frac{1}{2}\left(\frac{zz'}{w_t^2}\right)^2 + \cdots\right) \qquad \text{for } w_t \gg w \end{aligned} \tag{4.13}$$

Assuming $\psi(z',0)$ is real function (up to constant phase), one can see that integration of (4.8) except common $\sqrt{\frac{1}{2\pi i w_t^2}}$ and $\text{Exp}\left[\frac{i}{2}\frac{z^2}{w_t^2}\right]$ is

$$\begin{aligned} &\int \psi(z',0)\left(1 - i\frac{zz'}{w_t^2} - \frac{1}{2}\left(\frac{zz'}{w_t^2}\right)^2 + \cdots\right)dz' \in \mathbb{R} \text{ if } \psi(z',0) \text{ is even} \\ &\int \psi(z',0)\left(1 - i\frac{zz'}{w_t^2} - \frac{1}{2}\left(\frac{zz'}{w_t^2}\right)^2 + \cdots\right)dz' \in \mathbb{I} \text{ if } \psi(z',0) \text{ is odd} \end{aligned} \tag{4.14}$$

Thus for long time limit constant phase difference $\pi/2$ or $3\pi/2$ independent of z develops between even and odd function, and also meaning of long time limit is significant; $w_t \gg w$ where w is typical width of $\psi(z',0)$.

Now let's show that (4.6) is valid for the many-body state described by

$\psi_0(z,t)$ and $\psi_1(z,t)$ which are determined from (4.8). Here we claim that if following conditions are satisfied, (4.6) is valid for $w_t \gg w$.

1. $\psi_0(z,0)$ is an even function of z and $\psi_0(z,0) \in \mathbb{R}$.
   $\psi_1(z,0)$ is an odd function of z and $\psi_1(z,0) \in \mathbb{R}$
2. $|C_l|$ follows (5.72), $C_l \in \mathbb{R}$ and $\mathrm{sgn}(C_l, C_{l+2}) = -1$.

Here definition of w is vague, (*temporarily*) define w as

$$\int_{-2w}^{2w} |\psi_0(z,0)|^2 > 0.95 \quad \text{and} \quad \int_{-3w}^{3w} |\psi_1(z,0)|^2 > 0.95 \tag{4.15}$$

For $w_t \gg w$, (4.13) becomes accurate and $\psi_0(z,t)$ and $\psi_1(z,t)$ are approximated as

$$\begin{aligned} \psi_i(z, w_t \gg w) \simeq & \sqrt{\frac{1}{2\pi i w_t^2}}\ \mathrm{Exp}\left[\frac{iz^2}{2w_t^2}\right] \\ & \times \int dz' \psi_i(z',0)\left(1 - i\frac{zz'}{w_t^2} - \frac{1}{2}\left(\frac{zz'}{w_t^2}\right)^2 + \cdots\right) \end{aligned} \tag{4.16}$$

Looking at (4.14), it seems that following $\tilde{\psi}_i(z,t)$ can play the same role as $\psi_{it}(z)$ in (4.6).

$$\begin{aligned} \tilde{\psi}_0(z,t) &\equiv \int dz' \psi_0(z',0) \mathrm{Exp}\left[-i\frac{zz'}{w_t^2}\right] \\ &= \int dz' \psi_0(z',0)\left(1 - \frac{1}{2}\left(\frac{zz'}{w_t^2}\right)^2 + \cdots\right) \in \mathbb{R} \\ \tilde{\psi}_1(z,t) &\equiv -i\int dz' \psi_1(z',0) \mathrm{Exp}\left[-i\frac{zz'}{w_t^2}\right] \\ &= -i\int dz' \psi_1(z',0)\left(-i\frac{zz'}{w_t^2} + \frac{i}{6}\left(\frac{zz'}{w_t^2}\right)^3 + \cdots\right) \in \mathbb{R} \end{aligned} \tag{4.17}$$

Then

$$\begin{aligned} \psi_0(z, w_t \gg w) &\simeq \sqrt{\frac{1}{2\pi i w_t^2}}\ \mathrm{Exp}\left[\frac{iz^2}{2w_t^2}\right]\ \tilde{\psi}_0(z,t) \\ \psi_1(z, w_t \gg w) &\simeq i\sqrt{\frac{1}{2\pi i w_t^2}}\ \mathrm{Exp}\left[\frac{iz^2}{2w_t^2}\right]\ \tilde{\psi}_1(z,t) \end{aligned} \tag{4.18}$$

By taking $\psi_i(z,t)$ from (4.8), $\langle \hat{\psi}^\dagger(z,t)\hat{\psi}^\dagger(z',t)\hat{\psi}(z',t)\hat{\psi}(z,t)\rangle$ is written as follows with time independent TPDM elements $\langle \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l \rangle$ $(i, j, k, l = 0, 1)$.

$$
\begin{aligned}
&\langle \hat{\psi}^\dagger(z,t)\hat{\psi}^\dagger(z',t)\hat{\psi}(z',t)\hat{\psi}(z,t)\rangle = \\
&\qquad \langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0\rangle |\psi_0(z,t)|^2|\psi_0(z',t)|^2 + \langle \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1\rangle |\psi_1(z,t)|^2|\psi_1(z',t)|^2 \\
&\quad + \langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0\rangle \big(|\psi_0(z,t)|^2|\psi_1(z',t)|^2 + |\psi_0(z',t)|^2|\psi_1(z,t)|^2 \\
&\quad + \psi_0^*(z,t)\psi_1^*(z',t)\psi_1(z,t)\psi_0(z',t) + \psi_0^*(z',t)\psi_1^*(z,t)\psi_1(z',t)\psi_0(z,t)\big) \\
&\quad + \Big(\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1\rangle \psi_0^*(z,t)\psi_0^*(z',t)\psi_1(z',t)\psi_1(z,t) + (\mathrm{h.c.})\Big) \\
&\quad + \Big[\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_1\rangle (|\psi_0(z',t)|^2\psi_0^*(z,t)\psi_1(z,t) \\
&\quad + |\psi_0(z,t)|^2\psi_0^*(z',t)\psi_1(z',t)) + (\mathrm{h.c.})\Big] + \Big[0 \Leftrightarrow 1\Big]
\end{aligned}
\tag{4.19}
$$

(5.72) and second condition gives TPDM elements up to order of N^2 as same as in (4.1). Together with (4.18), density-density corelation becomes

$$
\begin{aligned}
&\simeq \Big(\bar{N}_0|\tilde{\psi}_0(z,t)|^2 + \bar{N}_1|\tilde{\psi}_1(z,t)|^2\Big)\Big(\bar{N}_0|\tilde{\psi}_0(z',t)|^2 + \bar{N}_1|\tilde{\psi}_1(z',t)|^2\Big) \\
&\qquad + 4\bar{N}_0\bar{N}_1\tilde{\psi}_0(z,t)\tilde{\psi}_0(z',t)\tilde{\psi}_1(z,t)\tilde{\psi}_1(z',t)
\end{aligned}
\tag{4.20}
$$

for $w_t \gg w$. And $\mathcal{R}(z,t)$ is,

$$
\begin{aligned}
\mathcal{R}(z,t) &\rightarrow \frac{(|\bar{N}_0\tilde{\psi}_0(z,t)|^2 + \bar{N}_1|\tilde{\psi}_1(z,t)|^2)^2 - 4\bar{N}_0\bar{N}_1|\tilde{\psi}_0(z,t)|^2|\tilde{\psi}_1(z,t)|^2}{(|\bar{N}_0\tilde{\psi}_0(z,t)|^2 + \bar{N}_1|\tilde{\psi}_1(z,t)|^2)^2 + 4\bar{N}_0\bar{N}_1|\tilde{\psi}_0(z,t)|^2|\tilde{\psi}_1(z,t)|^2} \\
&= \frac{\Big(|\tilde{\psi}_0(z,t)|^2\bar{N}_0 - |\tilde{\psi}_1(z,t)|^2\bar{N}_1\Big)^2}{(|\bar{N}_0\tilde{\psi}_0(z,t)|^2 + \bar{N}_1|\tilde{\psi}_1(z,t)|^2)^2 + 4\bar{N}_0\bar{N}_1|\tilde{\psi}_0(z,t)|^2|\tilde{\psi}_1(z,t)|^2}
\end{aligned}
\tag{4.21}
$$

Getting the same expression as in (4.6) finally. It is easy to show that $\mathcal{R}(z, t = 0) = 1$ for all z in contrast to long time limit. In addition, along $z' = -\lambda z$ $(\lambda > 0)$

$$
\begin{aligned}
&\mathcal{R}(z,t,z' = -\lambda z) = \\
&\frac{\Big(\bar{N}_0|\tilde{\psi}_0(z,t)|^2 - \bar{N}_1|\tilde{\psi}_1(z,t)|^2\Big)\Big(\bar{N}_0|\tilde{\psi}_0(\lambda z,t)|^2 - \bar{N}_1|\tilde{\psi}_1(\lambda z,t)|^2\Big)}{(|\bar{N}_0\tilde{\psi}_0(z,t)|^2 + \bar{N}_1|\tilde{\psi}_1(z,t)|^2)^2 + 4\bar{N}_0\bar{N}_1|\tilde{\psi}_0(z,t)|^2|\tilde{\psi}_1(z,t)|^2} \\
&+ \frac{2\bar{N}_0\bar{N}_1\Big(\tilde{\psi}_0(z,t)\tilde{\psi}_1(\lambda z,t) - \tilde{\psi}_0(\lambda z,t)\tilde{\psi}_1(z,t)\Big)^2}{(|\bar{N}_0\tilde{\psi}_0(z,t)|^2 + \bar{N}_1|\tilde{\psi}_1(z,t)|^2)^2 + 4\bar{N}_0\bar{N}_1|\tilde{\psi}_0(z,t)|^2|\tilde{\psi}_1(z,t)|^2}
\end{aligned}
\tag{4.22}
$$

In conclusion, it is shown that if there are two real initial wavefunction of similar size one is even and another is odd, an relative phase between them becomes independent of z as $w_t \gg w$ for TOF time evolution. And thus (4.21) can be applied not only for ground and 1st excited states of simple harmonic oscillator but also similarly for variety of even and odd orbitals pair in two-mode approximation for long time limit.

4.4 Comparison with Double Well Fragmentation

To contrast our result for density-density correlations in a single trap with the well-known result for a double well [26], for complete and self-contained discussion we briefly elaborate below on the latter. It is again noted that to avoid confusing this rather trivial fragmented state with the fragmented state of interest, in this thesis 'fragmented state' is not double well fragmented state unless double well is explicitly written after fragmented state.

A fragmented double well configuration describes independent BECs, i.e. simple Fock states of particle number N_{L} and N_{R}, respectively. It is known that after TOF expansion, there exists interference pattern coming from Hanburry-Brown-Twiss (HBT) correlation formated after two clouds 'meet' to each other. The orbitals $\psi_{\mathrm{L}}(z)$ and $\psi_{\mathrm{R}}(z)$ centers are displaced relative to each other by a distance d due to a repulsive barrier. The correlation functions are given by

$$\begin{aligned}
\left\langle \hat{\psi}^\dagger(z)\hat{\psi}(z)\right\rangle &= N_{\mathrm{L}}|\psi_{\mathrm{L}}(z)|^2 + N_{\mathrm{R}}|\psi_{\mathrm{R}}(z)|^2 \\
\left\langle \hat{\psi}^\dagger(z)\hat{\psi}^\dagger(z')\hat{\psi}(z')\hat{\psi}(z)\right\rangle &\simeq \left(N_{\mathrm{L}}|\psi_{\mathrm{L}}(z)|^2 + N_{\mathrm{R}}|\psi_{\mathrm{R}}(z)|^2\right) \\
&\quad \times \left(N_{\mathrm{L}}|\psi_{\mathrm{L}}(z')|^2 + N_{\mathrm{R}}|\psi_{\mathrm{R}}(z')|^2\right) \\
&\quad + 2N_{\mathrm{L}}N_{\mathrm{R}}\Re\left[\psi^*_{\mathrm{L}}(z)\psi_{\mathrm{L}}(z')\psi^*_{\mathrm{R}}(z')\psi_{\mathrm{R}}(z)\right].
\end{aligned} \tag{4.23}$$

Here, $\psi_{\mathrm{L}}(z)$ and $\psi_{\mathrm{R}}(z)$ are chosen to be two Gaussians of width $w = \sqrt{1/\omega} \equiv 1$ each centered at $z = -d/2$, $z = d/2$ and defined as follows [27]

$$\psi_{\mathrm{L}}(z) = \frac{1}{\pi^{1/4}} \exp\left[-\frac{\left(z-\frac{d}{2}\right)^2}{2}\right], \quad \psi_{\mathrm{R}}(z) = \frac{1}{\pi^{1/4}} \exp\left[-\frac{\left(z+\frac{d}{2}\right)^2}{2}\right]. \tag{4.24}$$

Applying the noninteracting propagator to the initial orbitals as in Eq. (4.8), the time evolution of each Gaussian under TOF can be described by $e^{i\phi(z+d/2,t)}\tilde{\psi}(z+$

$d/2,t)$, $e^{i\phi(z-d/2,t)}\tilde{\psi}(z-d/2,t)$ where $\tilde{\psi}(z,t)$ and $\phi(z,t)$ are

$$
\begin{aligned}
\tilde{\psi}(z,t) &= \frac{1}{\pi^{1/4}(1+w_t^4)^{1/4}}\,\exp\left[-\frac{\tilde{z}^2}{2}\right],\\
\tilde{z} &= \frac{z}{\sqrt{1+w_t^4}}, \quad \phi(z,t) = \frac{1}{2t}\frac{t^2}{1+t^2}z^2 - \frac{3\pi}{4}.
\end{aligned}
\tag{4.25}
$$

For $\langle\hat{\psi}^\dagger(z,t)\hat{\psi}(z,t)\rangle$, this leads to

$$
\langle\hat{\psi}^\dagger(z,t)\hat{\psi}(z,t)\rangle = N_{\mathrm{L}}|\tilde{\psi}(z-d/2),t|^2 + N_{\mathrm{R}}|\tilde{\psi}(z+d/2),t|^2. \tag{4.26}
$$

The expected average of density in many experimental runs is just a Gaussian profile with normalization given by the total number of particles $N_{\mathrm{L}}+N_{\mathrm{R}}$.

On the other hand, the density-density correlation function furnishes non-trivial features, in form of HBT correlations, for which the above defined phase factor $\phi(z,t)$ plays the major role [26]

$$
\begin{aligned}
&\langle\hat{\psi}^\dagger(z,t)\hat{\psi}^\dagger(z',t)\hat{\psi}(z',t)\hat{\psi}(z,t)\rangle\\
&\simeq \left(N_{\mathrm{L}}|\tilde{\psi}(z-d/2,t)|^2 + N_{\mathrm{R}}|\tilde{\psi}(z+d/2,t)|^2\right)\\
&\times\left(N_{\mathrm{L}}|\tilde{\psi}(z'-d/2,t)|^2 + N_{\mathrm{R}}|\tilde{\psi}(z'+d/2,t)|^2\right)\\
&+2N_{\mathrm{L}}N_{\mathrm{R}}\left[\tilde{\psi}(z-d/2,t)\tilde{\psi}(z'-d/2,t)\tilde{\psi}(z'+d/2,t)\tilde{\psi}(z+d/2,t)\right]\\
&\times\cos\left(\frac{1}{t}\frac{t^2}{1+t^2}(z-z')d\right).
\end{aligned}
\tag{4.27}
$$

For $t \gg 1$, the HBT term becomes

$$
\begin{aligned}
&2N_{\mathrm{L}}N_{\mathrm{R}}\left[\tilde{\psi}(z-d/2,t)\tilde{\psi}(z'-d/2,t)\tilde{\psi}(z'+d/2,t)\tilde{\psi}(z+d/2,t)\right]\\
&\times\cos\left[d(\tilde{z}-\tilde{z}')\right].
\end{aligned}
\tag{4.28}
$$

The term in square brackets reduces to $\simeq |\tilde{\psi}(z)|^2|\tilde{\psi}(z')|^2$ as $\sqrt{1+w_t^4}\simeq t\gg d$. Looking at the cosine part, $(\tilde{z}-\tilde{z'})$ is scale-invariant, thus the initial d determines the correlation oscillation features in the long time limit. For $t\gg d$ and $t\gg 1$, we then have approximately

$$
\begin{aligned}
&\langle\hat{\psi}^\dagger(z,t)\hat{\psi}^\dagger(z',t)\hat{\psi}(z',t)\hat{\psi}(z,t)\rangle\\
&\simeq |\tilde{\psi}(z,t)|^2|\tilde{\psi}(z',t)|^2\left[N_{\mathrm{L}}^2+N_{\mathrm{R}}^2+2N_{\mathrm{L}}N_{\mathrm{R}}\left(1+\cos\left[d(\tilde{z}-\tilde{z}')\right]\right)\right].
\end{aligned}
\tag{4.29}
$$

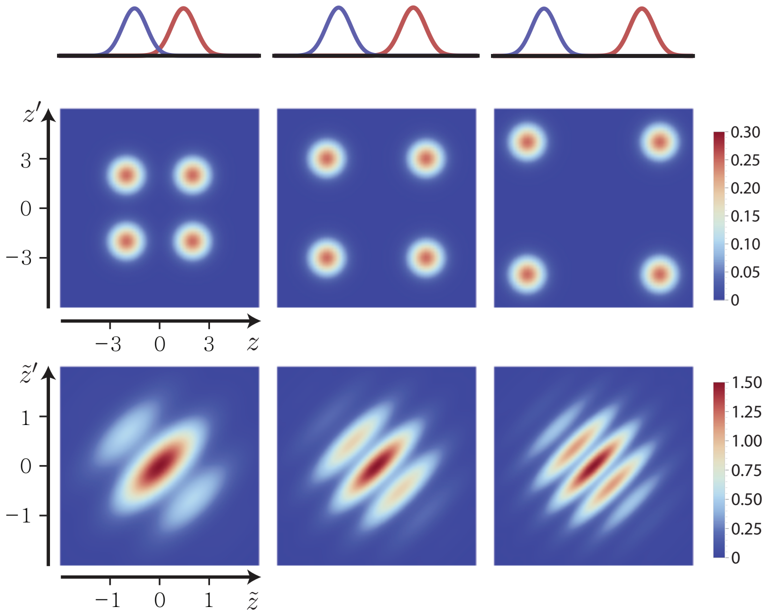

Figure 4.8: Density-density correlations of a symmetric double well fragmented state ($N_\mathrm{L} = N_\mathrm{R}$) before (top) and after (bottom) TOF. The correlation unit is $N^2/[\pi(1 + w_t^4)]$ [22].

In Fig. 4.8, we plot the correlations before and after TOF for separations $d = 4, 6, 8$, illustrating development of fringes in the off-diagonal direction $z' = -z$. One should compare these plots with those shown in Fig. 4.3 and Fig. 4.4.: in a single-trap fragmented state, there are no such density-density-correlation interference fringes to be detected. There will be further comparison between two different fragmented states in 5.3.

These considerations can be extended to, e.g., triple wells, which show qualitatively very similar correlation features. The basic differences in the correlation signal between single-trap and multi-well configurations are therefore not related to the number of maxima in the total density.

Chapter 5

Phase State Formalism

In this chapter, phase state basis is considered to interpret peculiar fluctuation in single-trap fragmented state. Phase state, overcomplete basis of two-mode states, applied to independent two BECs is described briefly to focus on strength of phase state formalism. Then condition to utilize phase state formalism for single-trap fragmented state is investigated. At the end of this chapter, it is shown that how phase state basis describes single-trap fragmented state in rather simple manner. At the same time, origin of large fluctuation in density-density correlation is explained.

5.1 Interference between Independent Two BECs

Interference between two independent, or phase uncorrelated, BECs has been dealt in several references, e.g. [3, 26]. Here it is dealt with a little bit different point of view, which is related to main question in utilizing phase states to express many-body state, especially two-mode case.

Consider two indepedent BECs each containing N_1, N_2 particles ($N_1+N_2 = N$ where N is total particle number of the system)

$$|N_1, N_2\rangle = \frac{(\hat{a}_1^\dagger)^{N_1}}{\sqrt{N_1!}}\frac{(\hat{a}_2^\dagger)^{N_2}}{\sqrt{N_2!}}\,|0\rangle\,. \tag{5.1}$$

Here it is noted that indices 1,2 are used for this two-mode state instead of 0, 1 temporarily. Where $\hat{a}_1^\dagger$, $\hat{a}_2^\dagger$ corresponds to normalized wavefunction describing each condensate and set to be orthogonal to each other. Using the fact that

$x^a y^b$ is component of $(x+y)^{a+b}$ together with $\int_0^{2\pi} d\phi e^{i(c-c')\phi} = 2\pi\delta_{c,c'}$,

$$
\begin{aligned}
|N_1, N_2\rangle &= \frac{(\hat{a}_1^\dagger)^{N_1}}{\sqrt{N_1!}} \frac{(\hat{a}_2^\dagger)^{N_2}}{\sqrt{N_2!}} |0\rangle \\
&\equiv C_{N_1,N_2} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \int_0^{2\pi} \frac{d\phi_2}{2\pi}\ e^{-(iN_1\phi_1 + iN_2\phi_2)} |\phi_1, \phi_2, N\rangle, \\
|\phi_1, \phi_2, N\rangle &= \frac{(\hat{\psi}^\dagger_{\phi_1,\phi_2})^N}{\sqrt{N!}} |0\rangle \equiv \frac{1}{\sqrt{N!}} \left(\frac{\hat{a}_1^\dagger e^{i\phi_1} + \hat{a}_2^\dagger e^{i\phi_2}}{\sqrt{2}} \right)^N |0\rangle
\end{aligned}
\tag{5.2}
$$

So arbitrary two-mode Fock state is equal to linear combination of phase state $|\phi_1, \phi_2, N\rangle$, thus any tow-mode state can be represented by l.c. of phase states. Knowledge of binomial distribution and short calculation gives C_{N_1,N_2} as follows,

$$
C_{N_1,N_2} = \sqrt{\frac{N_1!N_2!}{N!} 2^N} = \frac{1}{\sqrt{\frac{W(N_1,N_2)}{\sum_{N'=0}^{N} W(N',N-N')}}}
\tag{5.3}
$$

where $W(N_1, N_2) \equiv \frac{N!}{N_1!N_2!}$ and

$$
C_{N/2,N/2} = \sqrt{\frac{(N/2)!(N/2)!}{N!} 2^N} = \sqrt{\frac{N!!}{(N-1)!!}} \Longrightarrow \left(\frac{\pi N}{2}\right)^{1/4}
\tag{5.4}
$$

for large even N. Degree of freedom for 2 phases ϕ_1, ϕ_2 can be reduced into 1 phase ϕ which is related to 'relative' phase $\phi_1 - \phi_2$ directly. Integration over two phases can be simplified into one ϕ integration. By drawing out $e^{iN(\phi_1+\phi_2)/2}$ from $|\phi_1, \phi_2, N\rangle$, $N_1 + N_2 = N$ gives

$$
\begin{aligned}
|N_1, N_2\rangle &= C_{N_1,N_2} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \int_0^{2\pi} \frac{d\phi_2}{2\pi}\ e^{-(iN_1\phi_1 + iN_2\phi_2)} \frac{1}{\sqrt{N!}} |\phi_1, \phi_2, N\rangle \\
&= C_{N_1,N_2} \int_0^{2\pi} \frac{d\phi_1}{2\pi} \int_0^{2\pi} \frac{d\phi_2}{2\pi}\ e^{-i(N_1-N_2)(\phi_1-\phi_2)/2} \\
&\qquad \frac{1}{\sqrt{N!}} \left(\frac{\hat{a}_1^\dagger e^{i(\phi_1-\phi_2)/2} + \hat{a}_2^\dagger e^{-i(\phi_1-\phi_2)/2}}{\sqrt{2}} \right)^N |0\rangle
\end{aligned}
\tag{5.5}
$$

Now it is clear that only $\phi_1 - \phi_2$ contributes, and sum of two phases $\phi_1 + \phi_2$ is dummy degree of freedom.

Or, another sort of justification is possible. Many-body state lives in a ray space, thus multiplying constant complex number with unit magnitude to every states doesn't change physics. Simplifying the expression with such property, set the 'standard phase' with constraint; sum of phases is 0 in every phase state expression. $\phi \equiv \phi_1 = -\phi_2$ for two-mode case,

$$
\begin{aligned}
|N_1, N_2\rangle &= C_{N_1,N_2} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-i(N_1-N_2)\phi} |\phi, N\rangle \\
|\phi, N\rangle &= \frac{(\hat{\psi}_\phi^\dagger)^N}{\sqrt{N!}} |0\rangle \equiv \frac{1}{\sqrt{N!}} \left(\frac{\hat{a}_1^\dagger e^{i\phi} + \hat{a}_2^\dagger e^{-i\phi}}{\sqrt{2}} \right)^N |0\rangle
\end{aligned}
\tag{5.6}
$$

Again $\phi_1 + \phi_2$ is set to be 0, considered as irrelevant degree of freedom. Regarding this issue, in detail not schematic justification, it is well explained in [33] including dependence on $N_1 - N_2$.

Anyhow, any $|N_1, N_2\rangle$ can be cast into linear combination of phase states $|\phi, N = N_1 + N_2\rangle$, so phase states form at least overcomplete set of two-mode fixed N Hilbert space. Inner product between two different phase states is,

$$
\langle \phi', N | \phi, N \rangle = \left([\hat{\psi}_{\phi'}, \hat{\psi}_\phi^\dagger] \right)^N = Cos^N(\phi - \phi') \simeq \left(1 - \frac{(\phi - \phi')^2}{2} \right)^N \tag{5.7}
$$

where later approximation holds for small $\phi - \phi' \equiv \Delta\phi$. If $\Delta\phi$ is $\sim \frac{1}{\sqrt{N}}$ or larger, two phase states $|\phi, N\rangle$, $|\phi', N\rangle$ are orthogonal to each other. Orthogonality is ensured for relatively small N. $N = 100$, $\Delta\phi = \frac{2\pi}{10}$ gives inner product smaller than $1X10^{-8}$. This result, clearly shows that by dividing $\phi : [0, 2\pi]$ into $\sqrt{N}$ pieces one can obtain orthonormal set, but not complete set. Number of states needed to construct complete set of two-mode fixed N Hilbert space is N+1(running through from $N_1 = 0$ to $N_1 = N$). Since basis based on phase state hardly, or does not satisfy both orthonormal and complete condition, there is need of an argument on how effectively $\sqrt{N}$ basis describe physics of the system. So the point is that in which case this phase state is useful.

Applying idea of coarse-graining, one can consider $\sqrt{N}$ phase states as approximate complete set. For example, consider $|\Psi\rangle = \int_0^{2\pi} d\phi/2\pi \; C_\phi |\phi, N\rangle$ with $L \equiv \lfloor \sqrt{N} \rfloor$ (floor $\lfloor\rfloor$ takes integer part of contained number). Then redefine

$$
\int_{\frac{2(j-1)\pi}{L}}^{\frac{2j\pi}{L}} \frac{d\phi}{2\pi/L} C_\phi |\phi, N\rangle \equiv \left| \bar{\phi_j}, N \right\rangle, \quad j = 1, \cdots, L \tag{5.8}
$$

which gives $|\Psi\rangle = \sum_{j=1}^{L} \left| \bar{\phi_j}, N \right\rangle$. But $\left| \bar{\phi_j}, N \right\rangle$ is not-normalized, also vary depending on $|\Psi\rangle$, so $\left|\bar{\phi_j}, N\right\rangle$ is not good basis. Rather, introduce $\bar{C}_{\phi_j}$,

$$\int_{\frac{2(j-1)\pi}{L}}^{\frac{2j\pi}{L}} \frac{d\phi}{2\pi/L} C_\phi \equiv \bar{C}_{\phi_j}, \;\; |\Psi\rangle \approx \frac{1}{L}\sum_{j=1}^{L} \bar{C}_{\phi_j} |\phi_j, N\rangle = \sum_{j=1}^{L} \frac{\bar{C}_{\phi_j}}{L} \frac{(\hat{\psi}^\dagger_{\phi_j})^N}{\sqrt{N!}} |0\rangle \quad (5.9)$$

For example, consider $|N_1, N_2\rangle$. Then $\bar{C}_{\phi_j}$ is,

$$\bar{C}_{\phi_j} = \int_{\frac{2(j-1)\pi}{L}}^{\frac{2j\pi}{L}} \frac{d\phi}{2\pi/L} C_{N_1,N_2} e^{i(N_1-N_2)\phi} \quad (5.10)$$

Above simple averaging has analogy with interpreting mesoscopic system as set of 'representatives' consisted of accessible states sharing macroscopic variables. And this works in an simple case. When $|\Psi\rangle$ is Fock state $|N_1, N_2\rangle$ and $N_1 - N_2$ is small, i.e. $(N_1 - N_2)/\sqrt{N} \ll 1$, $C_\phi \sim e^{-i(N_1-N_2)\phi}$ is almost constant within integration interval. In this case averaging and 'discretizing' into $\sim \sqrt{N}$ basis is reasonable approach. To interpret the meaning of this process, let's have a look at expectation value of physical observable. In the case of Fock state with $N_1 = N_2$ for large N,

$$|N/2, N/2\rangle = \left(\frac{\pi N}{2}\right)^{1/4} \int_0^{2\pi} \frac{d\phi}{2\pi} |\phi, N\rangle \quad (5.11)$$

And, expectation value for density operator $\hat{\psi}^\dagger(z)\hat{\psi}(z)$ is written as,

$$\begin{aligned}
&\langle N/2, N/2| \hat{\psi}^\dagger(z)\hat{\psi}(z) |N/2, N/2\rangle \\
&= \left(\frac{\pi N}{2}\right)^{1/2} \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^{2\pi} \frac{d\phi'}{2\pi} \left\langle \phi', N\right| \hat{\psi}^\dagger(z)\hat{\psi}(z) |\phi, N\rangle \\
&= \left(\frac{\pi N}{2}\right)^{1/2} \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^{2\pi} \frac{d\phi'}{2\pi} \; \psi^*_{\phi'}(z)\psi_\phi(z) \left\langle \phi', N-1\right| \left. \phi, N-1\right\rangle, \qquad (5.12) \\
&\psi^*_{\phi'}(z)\psi_\phi(z) \\
&= \frac{1}{2}\left(|\psi_1(z)|^2 e^{i(\phi-\phi')} + |\psi_2(z)|^2 e^{-i(\phi-\phi')} + 2Re[\psi_1^*(z)\psi_2(z)\; e^{-i(\phi+\phi')}]\right)
\end{aligned}$$

Applying (5.9), then $\bar{C}_{\phi_j} = C_{N/2,N/2} = \left(\frac{\pi N}{2}\right)^{1/4}$ for $N_1 = N_2 = N/2$. Altogether

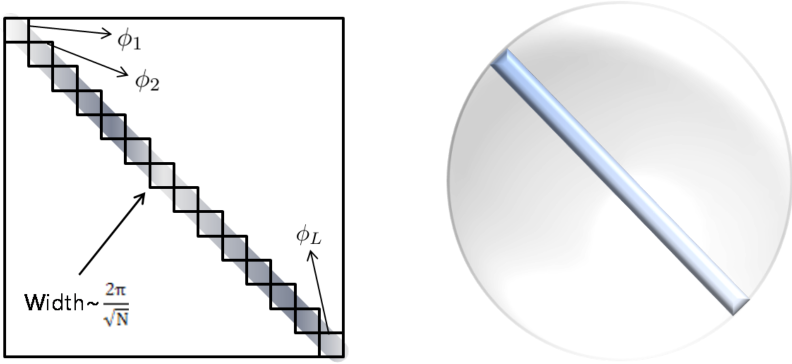

Figure 5.1: Visualization of block-diagonalization(Left) and matrix $\langle\phi'|\,\hat{\mathcal{O}}\,|\phi\rangle$(Right)

with orthonormality of $|\phi_j, N\rangle$,

$$
\begin{aligned}
&\langle N/2, N/2|\,\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\,|N/2, N/2\rangle \\
&\approx \sum_{j=1}^{L} \frac{1}{L^2}\left(\frac{\pi N}{2}\right)^{1/2} \langle\phi_j, N|\,\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\,|\phi_j, N\rangle \\
&= \sum_{j=1}^{L} \frac{1}{2L^2}\left(\frac{\pi N}{2}\right)^{1/2} \Big(|\psi_1(\mathbf{r})|^2 + |\psi_2(\mathbf{r})|^2 + 2Re[\psi_1^*(\mathbf{r})\psi_2(\mathbf{r})\; e^{-2i\phi}]\Big)
\end{aligned}
\tag{5.13}
$$

Which includes terms responsible for interference pattern, but equally probably for all ϕ_j. This amounts to block-diagonalization of continuous matrix $\langle\phi', N|\;\phi, N\rangle$ and may possibly be visualized as in Fig.5.1. The reason why this orthonormality is poweful is that, within the region where such averaged basis works, every physical observables can be represented as diagonal matrix each with definite phase ϕ. So time-evolution(especially useful for ballistic expansion of BECs) or statistical properties(like Hanbury-Twiss term as we will see) can be quite easily drawn out.

However, if not, i.e. when $N_1 - N_2 \sim \mathcal{O}(N)$, this time $C_\phi \sim e^{-i(N_1-N_2)\phi}$ oscillates rapidly resulting in $\bar{C}_{\phi_j} = 0$. So, one can conclude that this approach does not work at all in many practical case. Instead, focusing on matrix representation of physical observables, it is the point that whether $\langle\phi'|\,\hat{\mathcal{O}}\,|\phi\rangle$ can be utilized as trace of diagonal matrix, which is treated in the following.

By the way, generalization to many-mode case is straightforward. Consider

$\{\hat{a}_i^\dagger\}$ corresponding to each M orthonormal wavefunctions. Then,

$$\begin{aligned}
|N_1, \cdots, N_M\rangle &= \prod_{i=1}^{M} \frac{(\hat{a}_i^\dagger)^{N_i}}{\sqrt{N_i!}} |0\rangle \\
&= C_{N_1,\cdots,N_M} \left[\prod_{i=1}^{M} \int_0^{2\pi} \frac{d\phi_i}{2\pi} e^{-iN_i\phi_i}\right] \frac{1}{\sqrt{N!}} \left(\frac{\sum_{i=1}^{M} \hat{a}_i^\dagger e^{i\phi_i}}{\sqrt{M}}\right)^N |0\rangle \\
&\equiv C_{N_1,\cdots,N_M} \left[\prod_{i=1}^{M} \int_0^{2\pi} \frac{d\phi_i}{2\pi} e^{-iN_i\phi_i}\right] \left|\vec{\phi}, N\right\rangle
\end{aligned} \tag{5.14}$$

and

$$\begin{aligned}
&N_1 + \cdots + N_M \equiv N \\
&\left|\vec{\phi}, N\right\rangle = \frac{(\hat{\psi}_{\vec{\phi}}^\dagger)^N}{\sqrt{N!}} |0\rangle \equiv \frac{1}{\sqrt{N!}} \left(\frac{\sum_{i=1}^{M} \hat{a}_i^\dagger e^{i\phi_i}}{\sqrt{M}}\right)^N |0\rangle .
\end{aligned} \tag{5.15}$$

$C_{N_1,\cdots,N_M}$ is given as

$$\begin{aligned}
C_{N_1,\cdots,N_M} &= \sqrt{\frac{\prod_{i=1}^{M} N_i!}{N!} M^N} = \frac{1}{\sqrt{\frac{W(N_1,\cdots,N_M)}{\sum_{\{N_i'\}} W(N_1',\cdots,N_M')}}} \\
W(N_1, \cdots, N_M) &\equiv \frac{N_1! \cdots N_M!}{N!}.
\end{aligned} \tag{5.16}$$

One degree of freedom can be removed out from M phases, a similar process can be done with M-mode states as in two-mode state in principle therefore possible to express fixed number M-mode states as linear combination of phase states with degree of freedom of $M-1$ phases.

5.2 Phase State Basis and Diagonal Expression

Above approach yielded useful interpretation. But it is not effective to utilize phase state formalism for general two-mode states. Here we again discuss two BECs case shortly with [3, 26]. Then how to use phase state basis in our fragmented state is investigated, starting from discussion in [22] which elaborated basic idea in previous section. In the end, conditions to use 'diagonal expression' is stated, which is a little bit different from [22], and fragmented state in a single trap is interpreted. This time we again consider a single Fock state

$|N-l,l\rangle$, and define phase state $|\phi,N,N/2\rangle$ in a little bit different way from $|\phi,N\rangle$.

$$|N-l,l\rangle = \frac{(\hat{a}_0^\dagger)^{N-l}(\hat{a}_1^\dagger)^l}{\sqrt{(N-l)!l!}}|0\rangle = \int_0^{2\pi}\frac{d\phi}{2\pi}\sqrt{\frac{(N-l)!l!}{N!}2^N}\, e^{-il\phi}\,|\phi,N,N/2\rangle \tag{5.17}$$

with $|\phi,N,N/2\rangle$

$$|\phi,N,N/2\rangle = \frac{(\hat{\psi}^\dagger_{\phi,N,N/2})^N}{\sqrt{N!}}|0\rangle\,,\ \ \hat{\psi}^\dagger_{\phi,N,N/2} = \frac{\hat{a}_0^\dagger + e^{i\phi}\hat{a}_1^\dagger}{\sqrt{2}} \tag{5.18}$$

Here ϕ is relative phase between mode 0 and mode 1, which is half of ϕ in previous section. For detailed discussion on this can be found in [33].

$$\begin{aligned}
&\hat{\psi}(\boldsymbol{r})\,|\phi,N,N/2\rangle = \sqrt{N}\psi_{\phi,N,N/2}(\boldsymbol{r})\left|\phi,N-1,\frac{N-1}{2}\right\rangle\\
&\psi_{\phi,N,N/2}(\boldsymbol{r}) \equiv \left[\hat{\psi}(\boldsymbol{r}),\hat{\psi}^\dagger_{\phi,N,N/2}\right]\\
&\left\langle\phi',N,N/2\right|\phi,N,N/2\rangle = \exp\left[i\frac{N(\phi-\phi')}{2}\right]\left(\cos\left[\frac{\phi-\phi'}{2}\right]\right)^N.
\end{aligned} \tag{5.19}$$

In terms of $|\phi,N,N/2\rangle$, the expectation value of the density, $\hat{\rho}(\boldsymbol{r}) = \hat{\psi}^\dagger(\boldsymbol{r})\hat{\psi}(\boldsymbol{r})$ can be written as a double integration over two phase angles ϕ and ϕ'

$$\begin{aligned}
&\langle N-l,l|\,\hat{\rho}(\boldsymbol{r})\,|N-l,l\rangle = \frac{(N-l)!l!}{N!}2^N\\
&\times\int_0^{2\pi}\frac{d\phi'}{2\pi}\int_0^{2\pi}\frac{d\phi}{2\pi}\,e^{i(N-2l-1)\Delta\phi}(\cos\Delta\phi)^{N-1}\psi^*_{\phi',N,N/2}(\boldsymbol{r})\psi_{\phi,N,N/2}(\boldsymbol{r}),
\end{aligned} \tag{5.20}$$

where $\Delta\phi = (\phi-\phi')/2$.

In the large N limit, the N-th power of $\cos\Delta\phi$ is approximately $e^{-N(\Delta\phi)^2/2}$, with a value $\mathcal{O}(1)$ within the range $|\Delta\phi| < \pi/\sqrt{N}$. Thus we can safely reduce the double integral into an integral over the single phase ϕ by putting $\phi' \simeq \phi$ and approximate the exponential factor by unity provided $N-2l \ll \sqrt{N}$. For the case of the evenly distributed single Fock state, $l = N/2$ the following approximate equality is therefore obtained, cf. [3] chapter 13,

$$\langle N/2,N/2|\,\hat{\rho}(\boldsymbol{r})\,|N/2,N/2\rangle \simeq \int_0^{2\pi}\frac{d\phi}{2\pi}\,\langle\phi,N,N/2|\,\hat{\rho}(\boldsymbol{r})\,|\phi,N,N/2\rangle\,. \tag{5.21}$$

Thus a Fock state $|N/2, N/2\rangle$ can be interpreted as an ensemble of all phase (coherent) states $|\phi, N, l\rangle$ with equal probability [29]. This result is applicable not only for $\hat{\rho}(\boldsymbol{r})$ but also for any n-body operator $\hat{O}_n$ where $n \ll N$ when $N \to \infty$ [30]. That any ϕ measured with equal probability was experimentally shown with interference fringes resulting from the TOF overlap of two initially independent BECs. The offset of fringes was different for each experimental run [14]; this was later on confirmed for the interference of thirty condensates released from optical lattice wells [31]. Theoretically, the concept of phase states was previously applied to time-of-flight experiments for weakly depleted condensates [32], and for the measurement theory of many-body states (counting statistics) in [33].

For a general $|N-l, l\rangle$, when $l \neq N/2$, we redefine $\hat{\psi}^\dagger_{\phi,N,N/2}$ to $|\phi, N, l_0\rangle$ and $\psi_{\phi,N,N/2}(\boldsymbol{r})$ to $\psi_{\phi,N,l_0}(\boldsymbol{r})$ as follows [28, 33],

$$\begin{aligned}
&\hat{\psi}^\dagger_{\phi,N,l_0} = \frac{\sqrt{N-l_0}\hat{a}_0^\dagger + e^{i\phi}\sqrt{l_0}\hat{a}_1^\dagger}{\sqrt{N}}, \quad |\phi, N, l_0\rangle = \frac{(\hat{\psi}^\dagger_{\phi,N,l_0})^N}{\sqrt{N!}}\,|0\rangle \\
&\psi_{\phi,N,l_0}(\boldsymbol{r}) = \left[\hat{\psi}(\boldsymbol{r}), \hat{\psi}^\dagger_{\phi,N,l_0}\right].
\end{aligned} \tag{5.22}$$

We now calculate the expectation value of an arbitrary normal ordered n-body operator, which is rather essential with finite mode approximation. In calculating correlation function by expanding field operator in truncated finite number of modes, correlation function should be normal ordered. If not, correlation should be expressed as sum of normal ordered operators by commutation relation. See appendix **A** for details.

$$:\hat{O}_n: = :\prod_{i=1}^{n} \hat{\psi}^\dagger(\boldsymbol{r}_i)\hat{\psi}(\boldsymbol{r}_i):. \tag{5.23}$$

We are going to show that the expectation value of (5.23) can be computed in the form similar to (5.21) for a general two-mode many-body state $|\Psi\rangle = \sum_l C_l\,|N-l, l\rangle$.

For $|\Psi\rangle = \sum_l C_l\,|N-l, l\rangle$, we do not have an exact number state unless $C_l = \delta_{l,l_0}$. Therefore, we have to carefully select the appropriate l value to evaluate correlation functions in some given order. From (5.22), two-mode state

$|\Psi\rangle$ can be expressed in terms of $|\phi, N, l_0\rangle$ as

$$|\Psi\rangle = \sum_l C_l \, |N-l, l\rangle = \int_0^{2\pi} \frac{d\phi}{2\pi} \sum_l \mathcal{N}_{N,l_0;l} \, C_l e^{-il\phi} \, |\phi, N, l_0\rangle \, . \tag{5.24}$$

Looking at properties of phase state as basis, we would like to note that any value l_0 can be chosen as basis independent of given l on left hand side. So $|\phi, N, l_0\rangle$ with $\phi : 0 \sim 2\pi$ of any l_0 forms basis of two-mode state, and actually overcomplete set since

$$\left\langle \phi', N, l_0 \right| \left. \phi, N, l_0 \right\rangle = \left[\frac{(N-l_0) + e^{i(\phi-\phi')} l_0}{N} \right]^N \tag{5.25}$$

which has almost step-function like characteristic since near $|\phi - \phi'| \sim 1/\sqrt{N}$, value drops down rapidly from 1 to 0. In addition, for two different choices of l_0, say l_1 and l_2, inner product between $|\phi', N, l_1\rangle$ and $|\phi, N, l_2\rangle$ is

$$\left\langle \phi', N, l_1 \right| \left. \phi, N, l_2 \right\rangle = \left[\frac{\sqrt{(N-l_1)(N-l_2)} + \sqrt{l_1 l_2} e^{i(\phi-\phi')}}{N} \right]^N \tag{5.26}$$

and

$$\left| \left[\frac{\sqrt{(N-l_1)(N-l_2)} + \sqrt{l_1 l_2} e^{i(\phi-\phi')}}{N} \right] \right|^N \tag{5.27}$$

is less or equal than

$$\left| \left[\frac{\sqrt{(N-l_1)(N-l_2)} + \sqrt{l_1 l_2}}{N} \right] \right|^N \tag{5.28}$$

where (5.28) approaches to 0 as $|l_1 - l_2| \gg \sqrt{N}$. Therefore, here we suggest to apply following quasi orthogonality relation, since $\left\langle \phi', N, l_0 \right| \left. \phi, N, l_0 \right\rangle \simeq 0$ for $|\phi' - \phi| \gg 1/\sqrt{N}$ where $1/\sqrt{N}$ is tiny value for large N.

$$\left\langle \phi', N, l_0 \right| \left. \phi, N, l_0 \right\rangle \propto \delta(\phi - \phi') \tag{5.29}$$

and proportional constant can be determined from $\langle\Psi|\,\Psi\rangle = 1$ as follows.

$$1 = \int_0^{2\pi} \frac{d\phi'}{2\pi} \int_0^{2\pi} \frac{d\phi}{2\pi} \left\langle \phi', N, l_0 \right| C^*_{\phi'} C_\phi \left| \phi, N, l_0 \right\rangle \propto \int_0^{2\pi} \frac{d\phi}{2\pi} |C_\phi|^2 \tag{5.30}$$

and it can be inferred that $\propto \rightarrow =$ is achieved by putting $\int_0^{2\pi} \frac{d\phi}{2\pi}|C_\phi|^2 = 1$. Then, for any state $|\Psi\rangle$ we can have diagonal expression in ϕ as

$$\begin{aligned}
\langle\Psi|\,{:}\hat{O}{:}\,|\Psi\rangle &= \int_0^{2\pi} \frac{d\phi'}{2\pi} \int_0^{2\pi} \frac{d\phi}{2\pi} \left\langle \phi', N, l_0 \right| C^*_{\phi'} \,{:}\hat{O}{:}\, C_\phi \left| \phi, N, l_0 \right\rangle \\
&\propto \int_0^{2\pi} \frac{d\phi}{2\pi} |C_\phi|^2 \left\langle \phi, N, l_0 \right| {:}\hat{O}{:} \left| \phi, N, l_0 \right\rangle, \\
&\text{where } C_\phi = \sum_l \mathcal{N}_{N,l_0;l} C_l e^{-il\phi}
\end{aligned} \tag{5.31}$$

for any normal ordered correlation function ${:}\hat{O}{:}$. And $(\hat{\psi}(\mathbf{r}))^n \left| \phi, N, l_0 \right\rangle$ is

$$\left(\prod_{j=1}^{n} \sqrt{\frac{N-j+1}{N}} \right) \left(\sqrt{N-l_0}\,\psi_0(\mathbf{r}) + e^{i\phi}\sqrt{l_0}\,\psi_1(\mathbf{r}) \right)^n \frac{(\hat{\psi}^\dagger_{\phi,N,l_0})^{N-n}}{\sqrt{(N-n)!}} \left| 0 \right\rangle \tag{5.32}$$

where $\psi_i(\mathbf{r}) = [\hat{a}_i, \hat{\psi}^\dagger(\mathbf{r})]$ $(i = 0, 1)$ and $\psi(\mathbf{r})$, $\hat{a}_0$ and $\hat{a}_1$ are replaced by each $\sqrt{N-l_0}\psi_0(\mathbf{r}) + e^{i\phi}\sqrt{l_0}\psi_1(\mathbf{r})$ $\sqrt{N-l_0}$ and $e^{i\phi}\sqrt{l_0}$. In short, for n-th order normal ordered operator ${:}\hat{O}{:}$, $\left\langle \phi, N, l_0 \right| {:}\hat{O}{:} \left| \phi, N, l_0 \right\rangle$ is

$$\begin{aligned}
&\left\langle \phi, N, l_0 \right| {:}\hat{O}{:} \left(\hat{\psi}^\dagger(\mathbf{r}), \hat{\psi}(\mathbf{r}') \right) \left| \phi, N, l_0 \right\rangle \\
&= \left(\prod_{j=1}^{n} \frac{N-j+1}{N} \right) {:}\hat{O}{:} \left(\psi^*_{\phi,N,l_0}(\mathbf{r}), \psi_{\phi,N,l_0}(\mathbf{r}') \right) \\
&\left\langle \phi, N, l_0 \right| {:}\hat{O}{:} \left(\hat{a}^\dagger_0, \hat{a}^\dagger_1, \hat{a}_0, \hat{a}_1 \right) \left| \phi, N, l_0 \right\rangle \\
&= \left(\prod_{j=1}^{n} \frac{N-j+1}{N} \right) {:}\hat{O}{:} \left(\sqrt{N-l_0}, \sqrt{l_0}e^{-i\phi}, \sqrt{N-l_0}, \sqrt{l_0}e^{i\phi} \right)
\end{aligned} \tag{5.33}$$

where $\psi_{\phi,N,l_0} = \left[\hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger_{\phi,N,l_0} \right] = \psi_0(\mathbf{r}) + e^{i\phi}\psi_1(\mathbf{r})$. This not only gives a useful relation to calculate any correlation function from (5.33), but also infers that we can always express any correlation function as an *ensemble summation* over ϕ [14, 29, 31]. However, this approximation leads to a serious problem. For example we consider $:\hat{O}: = \hat{a}_0^\dagger \hat{a}_0$, then $\langle\Psi|\, \hat{a}_0^\dagger \hat{a}_0 \,|\Psi\rangle$ is proportional to

$$\int_0^{2\pi} \frac{d\phi}{2\pi} |C_\phi|^2 \langle \phi, N, l_0|\, \hat{a}_0^\dagger \hat{a}_0 \,|\phi, N, l_0\rangle = \int_0^{2\pi} \frac{d\phi}{2\pi} |C_\phi|^2 (N - l_0) \tag{5.34}$$

which gives $\langle\Psi|\, \hat{a}_0^\dagger \hat{a}_0 \,|\Psi\rangle = N - l_0$ then. For example, for the Fock state $|N-l, l\rangle = |N-l, l\rangle$ putting $l_0 = l$ then we have correct expression [3, 26]. But still result has two strange aspects. First, the value of l_0 depends on choice of basis therefore it should not affect the actual result but it does. Second, the result does not depends on C_ϕ therefore neither on C_l which is absurd. This means that simply applying (5.29) will not works at all. Here we present one more example to add one more reason why (5.29) does not hold simply and at the same to resolve this problem. If one writes $\langle\Psi|$ in terms of $\langle\phi, N, l_1|$ and writes $|\Psi\rangle$ in terms of $|\phi, N, l_2\rangle$ then

$$\langle\Psi|\, :\hat{O}: \,|\Psi\rangle \propto \int_0^{2\pi} \frac{d\phi}{2\pi} |C_\phi|^2 \langle \phi, N, l_1|\, :\hat{O}: \,|\phi, N, l_2\rangle = 0 \tag{5.35}$$

when $|l_1 - l_2|$ is large enough to make $\langle \phi, N, l_1|\, \phi, N, l_2\rangle \approx 0$, for example $|l_1 - l_2| \gg 1/\sqrt{N}$. This does not make a sense at all. Instead of applying (5.29) we carry out both ϕ and ϕ' integrations exactly for $\langle\Psi|\, \Psi\rangle$

$$\begin{aligned} \langle\Psi|\, \Psi\rangle = & \int_0^{2\pi} \frac{d\phi'}{2\pi} \int_0^{2\pi} \frac{d\phi}{2\pi} \\ & \left(\sum_{l'} \sum_{l} \mathcal{N}_{N,l_1;l'} \mathcal{N}_{N,l_2;l} C_{l'}^* C_l e^{il'\phi'} e^{-il\phi} \left\langle \phi', N, l_1 \right| \left. \phi, N, l_2 \right\rangle \right) \end{aligned} \tag{5.36}$$

and

$$
\begin{aligned}
&\left\langle \phi', N, l_1 \right| \phi, N, l_2 \rangle = \left[\hat{\psi}_{\phi',N,l_1}, \hat{\psi}^\dagger_{\phi,N,l_2} \right]^N \\
&= \sum_{j=0}^{N} \frac{N!}{(N-j)!j!} \frac{(\sqrt{(N-l_1)(N-l_2)})^{N-j}(\sqrt{l_1 l_2})^j}{N^N} e^{ij(\phi-\phi')}, \\
&\langle \Psi | \ \Psi \rangle \\
&= \sum_l \mathcal{N}_{N,l;l_1} \mathcal{N}_{N,l_2;l} |C_l|^2 \frac{N!}{(N-l)!l!} \frac{(\sqrt{(N-l_1)(N-l_2)})^{N-l}(\sqrt{l_1 l_2})^l}{N^N} = 1
\end{aligned}
\tag{5.37}
$$

which is a correct result. For $\hat{a}_0^\dagger \hat{a}_0$ and other correlation functions one can easily show with help of (5.32) that by similar calculation it recovers a correct expression in C_l after integrations over ϕ and ϕ' with arbitrary chosen phase state basis even with different l_1 and l_2 for bra and ket. These examples show that there exist condition(s) to have a diagonal form of $\langle \Psi | :\hat{O}: | \Psi \rangle$ in ϕ and if it is possible then proper choice of l_0 of phase state $|\phi, N, l_0\rangle$ is crucial depending on $|\Psi\rangle$. To summarize,

- By utilizing (5.29) one gets a diagonal expression in ϕ, but it does not work properly for general two-mode state $|\Psi\rangle = \sum_l C_l |N-l, l\rangle$.
- To resolve this, one can carry out integrations over ϕ and ϕ', then it just goes back to an expression with C_l again.
- Now a question to be answered is 'Can we determine condition(s) to use the diagonal expression in ϕ and which l_0 should we choose for phase state basis from C_l, which contains full information of two-mode state with given $\psi_0(\mathbf{r}), \psi_1(\mathbf{r})$?'.

The question can be rewritten as 'Can we find l_0 and C_ϕ which enables

$$
\langle \Psi | :\hat{O}: | \Psi \rangle = \int_0^{2\pi} \frac{d\phi}{2\pi} |C_\phi|^2 \langle \phi, N, l_0 | :\hat{O}: | \phi, N, l_0 \rangle \tag{5.38}
$$

for given C_l?'. To see whether there exist specific C_ϕ and $|\phi, N, l_0\rangle$ which enables (5.38), we write down $\langle \Psi | \ \Psi \rangle$ and $:\hat{O}: = \hat{a}_0^\dagger \hat{a}_0$ in terms of phase state basis $|\phi, N, l_0\rangle$ where value of l_0 will be determined in following calculations. First we write in phase state basis with (5.32) and (5.25) as

$$\begin{aligned}&\langle\Psi|\ \Psi\rangle\\&=\int_0^{2\pi}\frac{d\phi'}{2\pi}\int_0^{2\pi}\frac{d\phi}{2\pi}\sum_{l'=0}^{N}\sum_{l=0}^{N}\sum_{j=0}^{N}\frac{\mathcal{N}_{N,l_0;l'}\mathcal{N}_{N,l_0;l}}{(\mathcal{N}_{N,j;l_0})^2}C_{l'}^{*}C_l e^{i(l'-j)\phi'}e^{-i(l-j)\phi}\end{aligned}\tag{5.39}$$

$$\begin{aligned}\langle\Psi|\,\hat{a}_0^\dagger\hat{a}_0\,|\Psi\rangle=&\int_0^{2\pi}\frac{d\phi'}{2\pi}\int_0^{2\pi}\frac{d\phi}{2\pi}\sum_{l'=0}^{N}\sum_{l=0}^{N}\sum_{j=0}^{N-1}\mathcal{N}_{N,l_0;l'}\mathcal{N}_{N,l_0;l}\\&\times C_{l'}^{*}C_l e^{i(l'-j)\phi'}e^{-i(l-j)\phi}\frac{(N-l_0)^{N-j}l_0^j}{N^{N-1}}\frac{(N-1)!}{(N-1-j)!j!}\end{aligned}\tag{5.40}$$

where $\mathcal{N}_{N,l_0;l}=\sqrt{\frac{N^N}{(N-l_0)^{N-l}l_0^l}}\sqrt{\frac{(N-l)!l!}{N!}}$. And we integrate this over ϕ' which leads to $l'=j$ for both $\langle\Psi|\ \Psi\rangle$ and $\langle\Psi|\,\hat{a}_0^\dagger\hat{a}_0\,|\Psi\rangle$

$$\begin{aligned}\langle\Psi|\ \Psi\rangle&=\int_0^{2\pi}\frac{d\phi}{2\pi}\sum_{l'=0}^{N}\sum_{l=0}^{N}\frac{\mathcal{N}_{N,l_0;l}}{\mathcal{N}_{N,l_0;l'}}C_{l'}^{*}e^{il'\phi}C_l e^{-il\phi}\\\langle\Psi|\,\hat{a}_0^\dagger\hat{a}_0\,|\Psi\rangle&=\int_0^{2\pi}\frac{d\phi}{2\pi}\sum_{l'=0}^{N-1}\sum_{l=0}^{N}\frac{\mathcal{N}_{N,l_0;l}}{\mathcal{N}_{N,l_0;l'}}(N-l')C_{l'}^{*}e^{il'\phi}C_l e^{-il\phi}\end{aligned}\tag{5.41}$$

are obtained.

Now we write down $\langle\Psi|\ \Psi\rangle$ and $\langle\Psi|\,\hat{a}_0^\dagger\hat{a}_0\,|\Psi\rangle$ as (5.38) assuming there exist such C_ϕ. Then it should be like

$$\langle\Psi|\ \Psi\rangle=\int_0^{2\pi}\frac{d\phi}{2\pi}|C_\phi|^2,\quad\langle\Psi|\,\hat{a}_0^\dagger\hat{a}_0\,|\Psi\rangle=\int_0^{2\pi}\frac{d\phi}{2\pi}\,(N-l_0)\,|C_\phi|^2\tag{5.42}$$

for certain C_ϕ given from C_l. Comparing two different expressions for $\langle\Psi|\ \Psi\rangle$ and $\langle\Psi|\,\hat{a}_0^\dagger\hat{a}_0\,|\Psi\rangle$ we know that

$$\int_0^{2\pi}\frac{d\phi}{2\pi}|C_\phi|^2=1, N-l_0=\langle\Psi|\,\hat{a}_0^\dagger\hat{a}_0\,|\Psi\rangle\tag{5.43}$$

if there exist C_ϕ and l_0 enable (5.38). Furthermore, without $\mathcal{N}_{N,l_0;l}/\mathcal{N}_{N,l_0;l'}$ we get $C_\phi = \sum_l C_l e^{-il\phi}$ which satisfies $\int_0^{2\pi} \frac{d\phi}{2\pi} |C_\phi|^2 = 1$ always where

$$\int_0^{2\pi} \frac{d\phi}{2\pi} \left(\sum_{l'} C_{l'}^* e^{il'\phi} \right) \left(\sum_l C_l e^{-il\phi} \right) = \sum_l |C_l|^2 = 1 \tag{5.44}$$

and it is obvious that from (5.41) we cannot extract C_ϕ with $\mathcal{N}_{N,l_0;l}/\mathcal{N}_{N,l_0;l'}$. Therefore an issue to be resolved is whether $\mathcal{N}_{N,l_0;l}/\mathcal{N}_{N,l_0;l'}$

$$\frac{\mathcal{N}_{N,l_0;l}}{\mathcal{N}_{N,l_0;l'}} = \sqrt{\frac{(N-l)!l!}{(N-l')!l'!}} \sqrt{\frac{(N-l_0)^{l-l'}}{l_0^{l-l'}}} \tag{5.45}$$

can be eliminated effectively in calculating $\langle \Psi | {:}\hat{O}{:} | \Psi \rangle$. It cannot be in general, but what we are interested in is a calculation of correlation function of finite order n. And this means as long as (5.45) is almost unity at $l = l' + n$ for l_0 and given N and C_l, it is safe to put

$$C_\phi = \sum_l C_l e^{-il\phi} \tag{5.46}$$

then we can get a condition on C_l to have l_0 which satisfies

$$\langle \Psi | {:}\hat{O}{:} | \Psi \rangle \simeq \int_0^{2\pi} \frac{d\phi}{2\pi} \langle \phi, N, l_0 | {:}\hat{O}{:} | \phi, N, l_0 \rangle \tag{5.47}$$

by examining for specific correlation functions. For example here let's calculate $\langle \hat{a}_0^\dagger \hat{a}_1 \rangle$, $\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 \rangle$, and $\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \rangle$ by exact calculation of expectation value with $|\Psi\rangle$ and (5.38) with $C_\phi = \sum_l C_l e^{il\phi}$ denoted with each subscript Ψ and ϕ. For $\langle \hat{a}_0^\dagger \hat{a}_1 \rangle$,

$$\begin{aligned} \langle \hat{a}_0^\dagger \hat{a}_1 \rangle_\Psi &= \sum_{l=1}^{N} C_{l-1}^* C_l \sqrt{(N-l+1)l} \\ \langle \hat{a}_0^\dagger \hat{a}_1 \rangle_\phi &= \int_0^{2\pi} \frac{d\phi}{2\pi} \sqrt{(N-l_0)l_0} |C_\phi|^2 e^{i\phi} = \sum_{l=1}^{N} C_{l-1}^* C_l \sqrt{(N-l_0)l_0} \end{aligned} \tag{5.48}$$

For $\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 \rangle$,

$$\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 \rangle_\Psi = \sum_{l=1}^{N-1} |C_l|^2 (N-l) l$$
$$\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 \rangle_\phi = \frac{N-1}{N} \int_0^{2\pi} \frac{d\phi}{2\pi} (N-l_0) l_0 |C_\phi|^2 = \frac{N-1}{N} \sum_{l=0}^{N} |C_l|^2 (N-l_0) l_0 \tag{5.49}$$

For $\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \rangle$,

$$\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \rangle_\Psi = \sum_{l=2}^{N} C_{l-2}^* C_l \sqrt{(N-l+1)(N-l+2)l(l-1)}$$
$$\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \rangle_\phi = \int_0^{2\pi} \frac{d\phi}{2\pi} (N-l_0) l_0 |C_\phi|^2 e^{2i\phi} = \frac{N-1}{N} \sum_{l=2}^{N} C_{l-2}^* C_l (N-l_0) l_0 \tag{5.50}$$

Ignoring some factors, e.g. $(N-1)/N$ coming from $\prod_{j=1}^{n}(N-j+1)/N$ for $n=2$, the condition to utilize (5.38) for certain C_l distribution can be written as

$$\sum_l C_{l-a}^* C_l f(\sqrt{N-l}, \sqrt{l}) = \sum_l C_{l-a}^* C_l f(\sqrt{N-l_0}, \sqrt{l_0}), \quad l_0 \equiv \sum_l |C_l|^2 l \tag{5.51}$$

where $f(\sqrt{N-l}, \sqrt{l})$ is polynomial of $\sqrt{N-l}, \sqrt{l}$ and a is an integer. Therefore this condition can be rewritten as 'Up to how high moment, mean value l_0 yields correct result' when we consider kind of probabilistic distribution C_l considering *phase* ϕ_l of $C_l = |C_l| e^{i\phi_l}$ also. If a is not so large so that $|C_{l-a}| \simeq |C_l|$, (5.51) becomes

$$\sum_l |C_l|^2 e^{i(\phi_l - \phi_{l-a})} f(\sqrt{N-l}, \sqrt{l}) = \sum_l |C_l|^2 e^{i(\phi_l - \phi_{l-a})} f(\sqrt{N-l_0}, \sqrt{l_0}). \tag{5.52}$$

We first examine (5.52) for constant ϕ_l, therefore only $|C_l|$ matters. Then,

$$\sum_l |C_l|^2 f(\sqrt{N-l}, \sqrt{l}) = \sum_l |C_l|^2 f(\sqrt{N-l_0}, \sqrt{l_0}). \tag{5.53}$$

now it is easy to determine for Gaussian or Gaussian-like $|C_l|$ distribution. For example, if a width of $|C_l|$ distribution is $\mathcal{O}(\sqrt{N})$ or less then this condition holds in general for not so small N value. And it is not so hard to examine. Now looking at phase of C_l, ϕ_l, it is not easy to write down general argument. Instead, we think of period of ϕ_l, call it T here ($\phi_l = \phi_{l+T}$). If $|C_l|$ slowly varies, so $|C_l| \simeq |C_{l+T}|$, and also $l \simeq l+T$, then it is possible to use (5.38) if $|C_l|$ satisfies (5.53) only. Because we can average out $e^{i(\phi_l - \phi l-a)}$ in (5.52) for period T. To sum up

- (5.51) is the condition to use (5.38) with $C_\phi = \sum_l C_l e^{-il\phi}$ and $l_0 = \sum_l |C_l|^2 l$.
- For slowly varing $|C_l|$ ($|C_{l-a}| \simeq |C_l|$) with small enough a, (5.51) reduces to(5.52).
- If phase of C_l has certain period T and that period is small so that $|C_l| \simeq |C_{l+T}|$ and also $l \simeq l+T$, then (5.53) is enough to determine whether we can use (5.38) or not.
- Even though ϕ_l does not have certain period, we can apply maximum frequency T_{max} in Fourier (period) spectrum of ϕ_l.

5.2.1 Application to General NPC State

Now let us show that fragmented state in a single trap, NPC state with Gaussian $|C_l|$ distribution, satisfy above conditions and can be revealed with diagonal expression in phase state basis. First we assume that in $|C_l|$ distribution C_l of even l and C_l of odd l both have the same weight, so literally we can approximate $|C_l|$ to Gaussian function $C(l)$ of standard deviation σ and mean l_0. We will discuss later on the case when C_l of even l and C_l of odd l are different. If Gaussian $|C_l|$ distribution has $l_0 \pm \sigma$ within $0 \sim N$ and N is large enough to approximate C_l to continuous $C(l)$ then it is possible to approximate expectation value of $f(N-l,l)$ which is polynomial of $\sqrt{N-l}, \sqrt{l}$ as

$$\sum_l f(N-l,l)|C_l|^2 \simeq \int_{-\infty}^{\infty} f(N-l,l)|C(l)|^2\, dl. \tag{5.54}$$

$\langle \hat{a}_0^\dagger \hat{a}_0 \rangle$,$\langle \hat{a}_1^\dagger \hat{a}_1 \rangle$,$\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 \rangle$,$\langle \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \rangle$ and $\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 \rangle$ are directly determined from above expression. We have

$$\begin{aligned} &\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 \rangle = (N - l_0)^2 - (N - l_0) + \sigma^2, \; \langle \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \rangle = l_0^2 - l_0 + \sigma^2 \\ &\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 \rangle = (N - l_0) l_0 - \sigma^2 \end{aligned} \tag{5.55}$$

with $\langle \hat{a}_0^\dagger \hat{a}_0 \rangle = N - l_0$,$\langle \hat{a}_1^\dagger \hat{a}_1 \rangle = l_0$.

$\langle \hat{a}_0^\dagger \hat{a}_1 \rangle$,$\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_1 \rangle$,$\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \rangle$ are related to

$$\sum_l f(N-l,l) C_l^* C_{l+1} \simeq \sum_l f(N-l,l) |C_l|^2 e^{i(\phi_{l+1} - \phi_l)} \tag{5.56}$$

as long as $|C_l| \simeq |C_{l-1}|$ and their Hermitian conjugates are related to

$$\sum_l f(N-l,l) C_{l+1}^* C_l \simeq \sum_l f(N-l,l) |C_l|^2 e^{-i(\phi_{l+1} - \phi_l)} \tag{5.57}$$

which is just complex conjugate. However, this time ϕ_l interrupts continuum expression a lot. But still, if ϕ_l has certain period T such that ϕ_{l+T} and $|C_l|$ slowly varies so $|C_l| \simeq |C_{l+T}|$, we can carry out summation over $e^{i(\phi_{l+1} - \phi_l)}$ separately from l summation as

$$\sum_l f(N-l,l) |C_l|^2 e^{i(\phi_{l+1} - \phi_l)} \simeq \frac{1}{T} \sum_{l=0}^{T-1} e^{i(\phi_{l+1} - \phi_l)} \sum_l f(N-l,l) |C_l|^2 \tag{5.58}$$

therefore again we can have

$$\sum_l f(N-l,l) C_l^* C_{l+1} \simeq \frac{1}{T} \sum_{l=0}^{T-1} e^{i(\phi_{l+1} - \phi_l)} \int_{-\infty}^{\infty} f(N-l,l) |C(l)|^2 \, dl. \tag{5.59}$$

For NPC state, we have $T = 4$ and $\mathrm{sgn}(C_l C_{l\pm 2}) = -1$ which yields following string of ϕ_l

$$\cdots, 0, \theta_k, \pi, \theta_k + \pi, 0, \theta_k, \pi, \theta_k + \pi, \cdots \tag{5.60}$$

where θ_k is relative phase different between even sector and odd sector defined from $\phi_1 - \phi_0$ with chosen (or fixed) $\phi_0 = 0$, therefore

$$\frac{1}{4} \sum_{l=0}^{3} e^{i(\phi_{l+1} - \phi_l)} = e^{i\theta_k} + e^{i(\pi - \theta_k)} + e^{i\theta_k} + e^{-i(\pi + \theta_k)} = i \sin\theta_k \tag{5.61}$$

Finally we have $\left\langle \hat{a}_0^\dagger \hat{a}_1 \right\rangle \simeq i\sqrt{(N-l_0)l_0}\sin\theta_k$ and

$$\begin{aligned}\left\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_1 \right\rangle &\simeq i\left[(N-l_0)\sqrt{(N-l_0)l_0}\sin\theta_k - \sigma^2\right]\\ \left\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \right\rangle &\simeq i\left[l_0\sqrt{(N-l_0)l_0}\sin\theta_k + \sigma^2\right]\end{aligned} \tag{5.62}$$

and $\simeq$ comes from the fact that $f(N-l,l)$ this time includes square root of $N-l$ or l. And we do not have clear relation between expectation values over $|C_l|^2$ of square root of $N-l$ or l and σ not like in the case of expectation value of $N-l$ or l.

Finally for $\left\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \right\rangle$ (an expectation value of $\left\langle \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_0 \hat{a}_0 \right\rangle$ is just given from complex conjugate) we have

$$\begin{aligned}&\sum_l \sqrt{(N-l+1)(N-l+2)l(l-1)}C_{l-2}^* C_l\\ &\simeq \frac{1}{T}\sum_{l=0}^{T-1} e^{i(\phi_{l+2}-\phi_l)}\int_{-\infty}^{\infty}\sqrt{(N-l)(N-l-1)(l+1)(l+2)}|C(l)|^2\,dl.\end{aligned} \tag{5.63}$$

And $\frac{1}{T}\sum_{l=0}^{T-1} e^{i(\phi_{l+2}-\phi_l)}$ is simply

$$\frac{1}{4}\sum_{l=0}^{3} e^{i(\phi_{l+2}-\phi_l)} = -1 \tag{5.64}$$

from $T=4$ and $\operatorname{sgn}(C_l C_{l\pm 2}) = -1$. Therefore we have

$$\left\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \right\rangle \simeq -(N-l_0)l_0 + \sqrt{(N-l_0)l_0} + \sigma^2 \tag{5.65}$$

For a while, it is worthwhile to point out how relative weight between C_l of even l and C_l of odd l affect $\left\langle \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l \right\rangle$ because for general NPC state we do not need to have exactly same relative weight between C_l of even l and C_l of odd l. Both C_l of even l and C_l of odd l has Gaussian distribution *of same σ and l_0* but with different magnitude of normalization as

$$\sum_{i=0,2,\cdots} |C_l|^2 = |c|^2, \sum_{i=1,3,\cdots} |C_l|^2 = |c|^2 u^2, \quad |c|^2(1+u^2) = 1. \tag{5.66}$$

For TPDM like $\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 \rangle$ we have

$$
\begin{aligned}
\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 \rangle &= \sum_{l=2}^{N} (N-l)(N-l-1)|C_l|^2 \\
&= \left(\sum_{i=0,2,\cdots} (N-l)(N-l-1)|C_l|^2 + \sum_{i=1,3,\cdots} (N-l)(N-l-1)|C_l|^2 \right).
\end{aligned} \tag{5.67}
$$

Since both C_l of even l and C_l of odd l has Gaussian distribution of same σ and l_0 irrelevant of the value of a and b we have the same result as $a = b$ except an error from continuum assumption further; we applied continuum assumption for $\Delta l = 1$ for $l = 0, 1, 2, \cdots$ but this time continuum assumption is applied for $\Delta l = 2$ each even sector of $l = 0, 2, \cdots$ and odd sector of $l = 1, 3, \cdots$. It is of the same for $\langle \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \rangle$ and $\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 \rangle$.

As regards to $\langle \hat{a}_0^\dagger \hat{a}_1 \rangle$,$\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_1 \rangle$,$\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \rangle$ we have summation over $C_{l+1}^* C_l$ which leads to

$$
\begin{aligned}
&\sum_l f(N-l,l) C_l^* C_{l+1} \simeq \frac{1}{T} \sum_{l=0}^{T-1} e^{i(\phi_{l+1}-\phi_l)} |c|^2 \\
&\times \left[\sum_{i=0,2,\cdots} f(N-l,l)|C_l||C_{l+1}| + u^2 \sum_{i=1,3,\cdots} f(N-l,l)|C_l||C_{l+1}| \right]
\end{aligned} \tag{5.68}
$$

where $|C_{l+1}| = u|C_l|$ for even l and $|C_{l+1}| = 1/u|C_l|$ for odd l. This lead to overall $|c|^2 u$ factor for different a and b. When $u = 1$ we have $|c| = 1/\sqrt{2}$, therefore we can conclude that values of $\langle \hat{a}_0^\dagger \hat{a}_1 \rangle$, $\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_1 \rangle$, $\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \rangle$ and their conjugates are $2|c|^2 u$ times multiplied compared to $|a| = |b|$ case. So we have $\langle \hat{a}_0^\dagger \hat{a}_1 \rangle \simeq 2|c|^2 ui \sqrt{(N-l_0)l_0} \sin\theta_k$ and

$$
\begin{aligned}
\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_1 \rangle &\simeq 2|c|^2 ui \left[(N-l_0)\sqrt{(N-l_0)l_0} \sin\theta_k - \sigma^2 \right] \\
\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \rangle &\simeq 2|c|^2 ui \left[l_0 \sqrt{(N-l_0)l_0} \sin\theta_k + \sigma^2 \right]
\end{aligned} \tag{5.69}
$$

We want to note that these results are consistent with equation (4) of [21] which already considered general NPC state and its property.

Finally for $\left\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \right\rangle$ we have summation over $C_{l-2}^* C_l$ which leads to

$$\begin{aligned} &\sum_{l=2} f(N-l,l) C_{l-2}^* C_l \simeq \frac{1}{T} \sum_{l=0}^{T-1} e^{i(\phi_{l+2}-\phi_l)} \\ &\left[|a|^2 \sum_{i=0,2,\cdots} f(N-l,l)|C_l||C_{l+2}| + |b|^2 \sum_{i=1,3,\cdots} f(N-l,l)|C_l||C_{l+2}| \right] \end{aligned} \tag{5.70}$$

here $l = 2$ is explicitly written under summation appears in first, not to make confusion and to clearly show that $l \to l+2$ is after $\simeq$. This time we can just apply $|C_l| \simeq |C_{l+2}|$ irrelevant of a and b so we know that values of a and b does not affect $\left\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \right\rangle$ and its conjugate. Therefore, it is shown that conditions above to use diagonal expression is fulfilled for single-trap fragmentation.

5.3 Revealing Single-Trap Fragmentation with the Phase State Basis

We now investigate the properties of the phase state amplitude C_ϕ. This is a discrete Fourier transform of C_l; thus we expect a canonical relation between C_ϕ and C_l, giving a Heisenberg indeterminacy relation of the form

$$\Delta|C_\phi|\Delta|C_l| \sim 1. \tag{5.71}$$

As an example, we will consider the continuum approximation for the two-mode Hamiltonian discussed in [19]. Then the magnitude $|C(l)|$ is a shifted Gaussian:

$$|C(l)| = \frac{1}{(\pi a^2)^{1/4}} \exp\left[-\frac{\left(l - \frac{N}{2} - \mathcal{S}\right)^2}{2a_{\rm osc}^2} \right]. \tag{5.72}$$

single-trap fragmented state has $\mathrm{sgn}(C_l, C_{l\pm2}) = -1$ in (3.22) [19], but here we fix the value of $a_{\mathbf{osc}}^2 = N\sqrt{2/3}$ choosing orbitals set in [8], and change the shift $\mathcal{S}$ to see the effect of degree of fragmentation $\mathcal{F}$ in (2.1). Fig. 5.2 shows two particular examples for the resulting C_ϕ distribution. The degree of fragmentation $\mathcal{F}$ does not affect the relative heights of the peaks in the distribution $|C_\phi|$ [34]. In Fig.5.3 we verify the expectation, based on (5.71), that the C_ϕ distribution becomes wider the smaller a is (and thus the more narrow the $|C_l|$ distribution).

For a fragmented condensate many-body state $|\Psi\rangle$ in the natural basis which can be expressed as a superposition of phase states, $|\Psi\rangle = \int d\phi \; C_\phi \, |\phi, N\rangle$, the condition $\left\langle \hat{a}_0^\dagger \hat{a}_1 \right\rangle = 0$ leads to

$$\int_0^{2\pi} d\phi \; |C_\phi|^2 e^{i\phi} = 0. \tag{5.73}$$

The corresponding C_ϕ distribution for the single-trap fragmented state has two peaks, at values of ϕ separated by π. They are symmetrically located at $\phi = \pi/2, \; 3\pi/2$ for fragmented state in a single trap, as can bee seen in Fig.5.2, Fig.5.3.

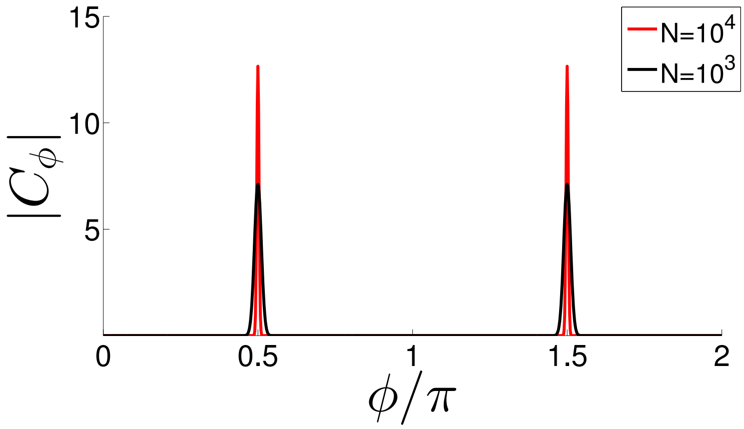

Figure 5.2: For maximal fragmentation $\mathcal{F} = 1$ ($\mathcal{S} = 0$), $N = 1000$, the modulus $|C_\phi|$ is centered at $\pi/2$ and $3\pi/2$ and the width $\Delta|C_\phi| \sim \pi/\sqrt{N} \simeq 0.1$. In red we show the distribution for $N = 10000$, all other parameters identical; then $\Delta|C_\phi| \sim \pi/\sqrt{N} \simeq 0.03$ [22].

The distribution of constant $|C_\phi|$ of a double well fragmented state in the left- and right-well basis obviously also satisfies (5.73). We now compare the two different types of fragmented state, double well and single-trap, by their density-density correlation function $\rho_2(z, z')$, using their respective C_ϕ distributions. Let us assume that we have a many-body state which can be described by a phase state distribution satisfying (5.73). For easy and direct comparison with the double well discussed in the existing section, we write the formulas below in one spatial dimension, noting that all results can be readily generalized to

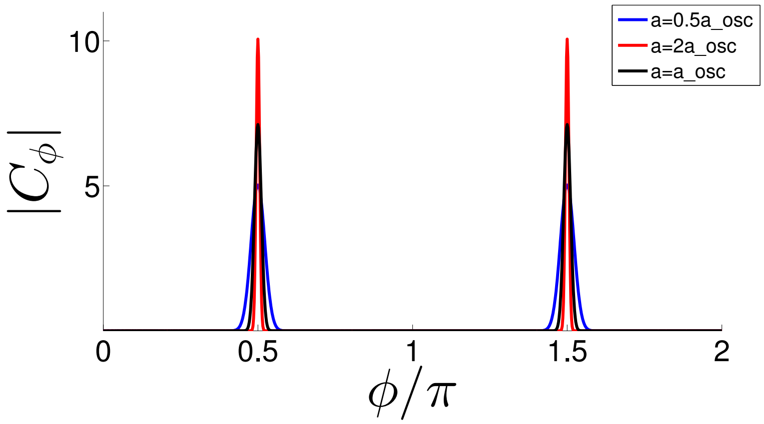

Figure 5.3: Variation of the width of the $|C_\phi|$ distribution upon increasing or decreasing the width a in the Gaussian amplitude distribution (5.72). All other parameters identical to Fig.5.2 [22].

arbitrary dimension. The density $\rho(z)$ is given as $\rho(z) = N_0|\psi_0(z)|^2 + N_1|\psi_1(z)|^2$ from $l_0 = N_1$ and (5.73). Therefore, $\rho(z)$ does not reveal any details of the C_ϕ distribution. For the second-order correlations, on the other hand, we have

$$\begin{aligned}
&\rho_2(z,z') \\
&= \int_0^{2\pi} \frac{d\phi}{2\pi}\, |C_\phi|^2 \left(\rho(z) + \sqrt{N_0N_1}\left(e^{i\phi}\psi_0^*(z)\psi_1(z) + e^{-i\phi}\psi_0(z)\psi_1^*(z)\right)\right) \\
&\times \left(z \to z'\right) \qquad (5.74) \\
&= \int_0^{2\pi} \frac{d\phi}{2\pi}\, |C_\phi|^2 \Big(\rho(z)\rho(z') \\
&+2N_0N_1\Re\left[\psi_0^*(z)\psi_1^*(z')\psi_0(z')\psi_1(z) + e^{2i\phi}\psi_0^*(z)\psi_0^*(z')\psi_1(z')\psi_1(z)\right]\Big),
\end{aligned}$$

where (5.73) is used in the second line. We now note that the integration of $|C_\phi|^2 e^{2i\phi}$ over ϕ can depend on details of the C_ϕ distribution. For double well fragmentation, $|C_\phi|$ is constant for all ϕ, so that $\rho_2(z,z')$ becomes

$$\rho_2(z,z') = \rho(z)\rho(z') + 2N_0N_1\Re\left[\psi_0^*(z)\psi_1^*(z')\psi_0(z')\psi_1(z)\right]. \qquad (5.75)$$

Thus the term $\propto e^{2i\phi}$ in the second line of (5.74) vanishes after integration, and only the HBT correlation term in Eq. (4.23) ($0 \to \mathrm{L}, 1 \to \mathrm{R}$) survives apart from the simple product of $\rho(z)$ and $\rho(z')$. Turning to the single-trap fragmented state, which has a C_ϕ distribution with two peaks at $\phi = \pi/2, 3\pi/2$, we obtain

$$\begin{aligned}
&\rho_2(z,z') = \rho(z)\rho(z') \\
&+ 2N_0N_1\Re\left[\psi_0^*(z)\psi_1^*(z')\psi_0(z')\psi_1(z) - \psi_0^*(z)\psi_0^*(z')\psi_1(z')\psi_1(z)\right].
\end{aligned} \tag{5.76}$$

The correlation function hence acquires a term directly related to negative pair coherence, distinct from HBT, which stems from the two-peak structure of the C_ϕ distribution.

5.3.1 Concluding Remark

We therefore conclude that the phase-state analysis distinguishes single-trap fragmented state from a double well fragmented state not only due to the absence of HBT terms in the density-density correlations, but also because of the existence of an additional term related to two peaks structure of $|C_\phi|$ distribution.

Further, single-trap fragmented state can be described with two phase states $|\pi/2, N, l_0\rangle$ and $|3\pi/2, N, l_0\rangle$ out of phase π with equal probabilistic weight. Fragmented state dealt here is pure state, not mixed state. Therefore, single-trap fragmented state $|\Psi\rangle$ can be described by

$$\frac{1}{\sqrt{2}}\Big(|\pi/2, N, l_0\rangle + e^{i\theta}|3\pi/2, N, l_0\rangle\Big) \tag{5.77}$$

which is superposition of $|\pi/2, N, l_0\rangle$ and $|3\pi/2, N, l_0\rangle$. This does not mean explicit equality $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\pi/2, N, l_0\rangle + e^{i\theta}|3\pi/2, N, l_0\rangle\right)$. However, correlation function is almost the same up to fifth order correlation function for $N = 1000, l_0 = 100$ [22], two states can be effectively considered as the same many-body state. This superposition of phase states then resembles superposition of coherent states, $|\alpha\rangle + e^{i\theta}|-\alpha\rangle$ since

1. Superposition of states with relative phase difference π.
2. $|\phi, N, l_0\rangle = \frac{(\hat{\psi}^\dagger_{\phi,N,l_0})^N}{\sqrt{N!}}|0\rangle$ approaches coherent state as $N \to \infty$ [11].

Approaching coherent state here means that $\hat{\psi}(\mathbf{r})|\phi, N, l_0\rangle \simeq \psi_{\phi,N,l_0}(\mathbf{r})|\phi, N, l_0\rangle$, which violates N conservation, becomes more exact as N gets larger.

And further, for general NPC state which is superposition of $|\mathrm{Even}\rangle$ and $|\mathrm{Odd}\rangle$, can be also expressed with $|\pi/2, N, l_0\rangle$ and $|3\pi/2, N, l_0\rangle$ but weight on two phase states could be different. Therefore there is possibility to have superposition of coherent states, cat state, with not equal probabilistic weight.

Chapter 6

Approximate Coherent State

Strict number conservation of phase state $|\phi, N, l_0\rangle$ is major obstacle in being coherent state where coherent state includes various number states with probability from Poisson distribution. Instead, from the fact that

$$\begin{aligned} |\phi, N, l_0\rangle &= \frac{1}{\sqrt{N!}}\left(\frac{\sqrt{N-l_0}\hat{a}_0^\dagger + e^{i\phi}\sqrt{l_0}\hat{a}_1^\dagger}{\sqrt{N}}\right)^N |0\rangle \\ &= \sum_{l=0}^{N}\sqrt{\frac{(N-l_0)^{N-l}l_0^l}{N^N}}\sqrt{\frac{N!}{(N-l)!l!}}\,|N-l, l\rangle \end{aligned} \tag{6.1}$$

gives probability for $|N-l, l\rangle$ proportional to multiplication of two Poisson distribution $\frac{(N-l_0)^{N-l}}{(N-l)!}, \frac{l_0^l}{l!}$. If l_0 gets smaller to $l_0 = 0$, probability to find l particles at mode 1 approaches that of actual coherent state; Poisson distribution. On the other hand, when l_0 approaches N, probability to find l particles at mode 0 approaches that of actual coherent state; again Poisson distribution. Therefore, by letting moving one particle from mode 0 to mode 1 to be creation of bosonic quasiparticle, or in opposite direction depending on the target many-body state, there is possibility to have approximate coherent state similar to phase state $|\phi, N, l_0\rangle$.

Here in this chapter, we construct approximate coherent state $|\beta\rangle$ and operator $\hat{b}, \hat{b}^\dagger$ based on the idea stated above following [36] with more details, after brief explanation about a superposition of coherent states. Accuracy of $|\beta\rangle$ and $\hat{b}, \hat{b}^\dagger$ as coherent state is investigated.

6.1 Superposition of Coherent States

We start from a commutation relation $[\hat{x}, \hat{p}] = i$. With a transformation of $\hat{x}, \hat{p}$ into bosonic annihilation operator $\hat{a} \equiv (\hat{x} + i\hat{p})/\sqrt{2}$ and bosonic creation operator $\hat{a}^\dagger \equiv (\hat{x} - i\hat{p})/\sqrt{2}$ one gets $[\hat{a}, \hat{a}^\dagger] = 1$. And it is possible to construct a number state $|N\rangle$ with vacuum state $|0\rangle$ defined as follows

$$|N\rangle \equiv \frac{(\hat{a}^\dagger)^N}{\sqrt{N!}} |0\rangle \,, \; \hat{a} |N\rangle = \sqrt{N} |N-1\rangle \,, \; \hat{a}^\dagger |N\rangle = \sqrt{N+1} |N+1\rangle \tag{6.2}$$

with $\hat{a}|0\rangle = 0$. Where a general state in number state basis can be written as $\sum_{n=0}^{\infty} C_n |n\rangle$, there exists an eigenstate of $\hat{a}$ of eigenvalue $\alpha = |\alpha| e^{i\phi_\alpha}$

$$|\alpha\rangle \equiv A_\alpha \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = A_\alpha e^{\alpha \hat{a}^\dagger} |0\rangle \,, \; A_\alpha = \left(\sum_{n=0}^{\infty} \frac{|\alpha|^2}{n!} \right)^{-\frac{1}{2}} = e^{-\frac{|\alpha|^2}{2}} \tag{6.3}$$

Furthermore, $D(\alpha)|0\rangle = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} |0\rangle$ is also the unit normalized eigenstate [35] of $\hat{a}$ with the eigenvalue α where unit normalization is easily shown from $(\alpha \hat{a}^\dagger - \alpha^* \hat{a})^\dagger = -(\alpha \hat{a}^\dagger - \alpha^* \hat{a})$. And two kinds of eigenstates are of the same which can be proved from reduced version of BCH (Baker-Campbell-Hausdorff) theorem where $[\hat{a}, \hat{a}^\dagger] = 1$ commutes with both $\hat{a}$ and $\hat{a}^\dagger$. Therefore $D(\alpha)|0\rangle = |\alpha\rangle$ and it is called as coherent state. Two coherent states $|\alpha_1\rangle$ and $|\alpha_2\rangle$ of different α_1 and α_2 are not orthogonal to each other since $\langle \alpha_1 | \alpha_2 \rangle$ is

$$e^{-\frac{1}{2}(|\alpha_1|^2 + |\alpha_2|^2 - 2\alpha_2^* \alpha_1)} = e^{-\frac{1}{2}(|\alpha_1|^2 - |\alpha_2|)^2} e^{-|\alpha_1||\alpha_2|(1 - \cos(\phi_{\alpha_1} - \phi_{\alpha_2}))} \tag{6.4}$$

thus $\langle \alpha_1 | \alpha_2 \rangle \neq \delta(\alpha_1 - \alpha_2)$. It is not a dirac-delta continuous complete basis as $|x\rangle$ and $|p\rangle$. But if $|\alpha_1| - |\alpha_2|$ is large, or $|\alpha_1||\alpha_2| \sin^2(\phi_{\alpha_1} - \phi_{\alpha_2})$ is large, still $\langle \alpha_1 | \alpha_2 \rangle$ approaches to 0. Here we'd like to note that it is easy to calculate an expectation value of any operator for coherent state. Since $\langle \alpha | \hat{a}^\dagger = \alpha^* \langle \alpha | \,, \; \hat{a}^\dagger |\alpha\rangle = \alpha |\alpha\rangle$, for any normal ordered operator $: \hat{O}(\hat{a}, \hat{a}^\dagger):$

$$\left\langle : \hat{O}(\hat{a}, \hat{a}^\dagger) : \right\rangle =: \hat{O}(\alpha, \alpha^*) : . \tag{6.5}$$

Therefore one can instantly get an expectation value evaluated for coherent state by contracting operator to a summation of normal ordered operators.

In quantum optics, an experimental realization of superposition of two coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ was done by click counting [37, 40, 41] or homodyne detection on photon number states [42]

$$\mathcal{N}(|\alpha\rangle + e^{i\theta}|-\alpha\rangle), \text{ where } \mathcal{N} = \frac{1}{\sqrt{2(1+\cos\theta e^{-2|\alpha|^2})}}. \tag{6.6}$$

When $\theta = 0$ it is called as an even state where only even photon number states are allowed, and when $\theta = \pi$ it is called as an odd state where only odd photon number states are allowed. An expectation value of $:\hat{O}(\hat{a}, \hat{a}^\dagger):$ of above superposition of coherent state is

$$\begin{aligned}\langle :\hat{O}(\hat{a}, \hat{a}^\dagger): \rangle = & \frac{1}{2(1+\cos\theta e^{-2|\alpha|^2})} \Big(:\hat{O}(\alpha, \alpha^*): + :\hat{O}(-\alpha, -\alpha^*): \\ & + e^{i\theta} e^{-2|\alpha|^2} :\hat{O}(-\alpha, \alpha^*): + e^{-i\theta} e^{-2|\alpha|^2} :\hat{O}(\alpha, -\alpha^*): \Big) .\end{aligned} \tag{6.7}$$

Qaudratures $\hat{x} = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger)$ and $\hat{p} = \frac{1}{\sqrt{2}i}(\hat{a} - \hat{a}^\dagger)$ and their second moments are

$$\begin{aligned}\Big\langle \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger) \Big\rangle &= \frac{\sin\theta e^{-2|\alpha|^2}}{1+\cos\theta e^{-2|\alpha|^2}} \sqrt{2}|\alpha| \sin\phi_\alpha \\ \Big\langle \frac{1}{\sqrt{2}i}(\hat{a} - \hat{a}^\dagger) \Big\rangle &= -\frac{\sin\theta e^{-2|\alpha|^2}}{1+\cos\theta e^{-2|\alpha|^2}} \sqrt{2}|\alpha| \cos\phi_\alpha \\ \Big\langle \frac{1}{2}(\hat{a} + \hat{a}^\dagger)^2 \Big\rangle &= 2|\alpha|^2 \cos^2\phi_\alpha + \frac{1}{2} - \frac{2|\alpha|^2 \cos\theta e^{-2|\alpha|^2}}{1+\cos\theta e^{-2|\alpha|^2}} \\ -\Big\langle \frac{1}{2}(\hat{a} - \hat{a}^\dagger)^2 \Big\rangle &= 2|\alpha|^2 \sin^2\phi_\alpha + \frac{1}{2} - \frac{2|\alpha|^2 \cos\theta e^{-2|\alpha|^2}}{1+\cos\theta e^{-2|\alpha|^2}}\end{aligned} \tag{6.8}$$

therefore showing huge fluctuation of order of mean photon number $\langle \hat{a}^\dagger \hat{a} \rangle = |\alpha|^2$ in second moments of position $\hat{x}$ or momentum $\hat{p}$ depending on the value of ϕ_α. In simple harmonic oscillator, $\phi_\alpha = 0$ corresponds to large fluctuation in position where $\phi_\alpha = \pi/2$ corresponds to large fluctuation in momentum.

6.2 Construction of $|\beta\rangle$ in Two-Mode System

General two-mode state $|\Psi\rangle$ can be written as

$$|\Psi\rangle = \sum_{l=0}^{N} C_l |N-l, l\rangle, \quad |N-l, l\rangle = \frac{(\hat{a}_0^\dagger)^{N-l}(\hat{a}_1^\dagger)^l}{\sqrt{(N-l)!l!}} |0\rangle, \quad \sum_{l=0}^{N} |C_l|^2 = 1 \tag{6.9}$$

Here we'd like to construct bosonic creation operator $\hat{b}^\dagger$ and annihilation operator $\hat{b}$ which takes one particle from mode 0 (mode 1) to mode 1 (mode 0), therefore $\hat{b}, \hat{b}^\dagger$ requires following conditions to be satisfied.

- $\hat{b}\,|N-l, l\rangle = \sqrt{l}\,|N-l+1, l-1\rangle$
- $\hat{b}^\dagger\,|N-l, l\rangle = \sqrt{l+1}\,|N-l-, l+1\rangle$
- $\langle\Psi|\,[\hat{b}, \hat{b}^\dagger]\,|\Psi\rangle = 1$

We propose following $\hat{b}, \hat{b}^\dagger$ with $\hat{N}_0 \equiv \hat{a}_0^\dagger \hat{a}_0$ as

$$\hat{b} \equiv \lim_{\epsilon\to 0} \frac{1}{\sqrt{\hat{N}_0 + \epsilon}} \hat{a}_0^\dagger \hat{a}_1, \quad \hat{b}^\dagger \equiv \lim_{\epsilon\to 0} \hat{a}_1^\dagger \hat{a}_0 \frac{1}{\sqrt{\hat{N}_0 + \epsilon}} \tag{6.10}$$

where ϵ is introduced to regularize singularity happens for $\hat{b}^\dagger\,|0, N\rangle$ without ϵ. Then acting two operators towards $|N-l, l\rangle$ gives

$$\hat{b}\,|N-l, l\rangle = \sqrt{l}\,|N-l+1, l-1\rangle\,, \quad \hat{b}^\dagger\,|N-l, l\rangle = \sqrt{l+1}\,|N-l-, l+1\rangle \tag{6.11}$$

as desired except $\hat{b}^\dagger\,|0, N\rangle = 0$ due to the fact that there is no particle left in mode 0 to be drawn up to mode 1. Therefore this can be understood as an inherited limitation due to finite N. Except $|0, N\rangle$ commutation relation $[\hat{b}, \hat{b}^\dagger]$ evaluated for any $|N-l, l\rangle$ gives 1 where $\langle N-l, l|\,[\hat{b}, \hat{b}^\dagger]\,|N-l', l'\rangle = 0$ for $l \neq l'$ when $l \neq N$. Hence a difference $1 - \langle\Psi|\,[\hat{b}, \hat{b}^\dagger]\,|\Psi\rangle$ occured by $|0, N\rangle$ is

$$1 - \langle\Psi|\,[\hat{b}, \hat{b}^\dagger]\,|\Psi\rangle = \sum_{l=0}^{N} |C_l|^2 - \left(\sum_{l=0}^{N-1} |C_l|^2 - N|C_N|^2\right) = (N+1)|C_N|^2 \tag{6.12}$$

thus as long as $|C_N| \ll 1/\sqrt{N}$, it is always possible to construct bosonic ladder operator for arbitrary two-mode system.

Then we can think this two-mode system of fixed N as a system with quasi vacuum $|\bar{0}\rangle \equiv |N, 0\rangle$ violating number conservation as photon does, but maximum number of such quasi particles is limited up to N with ladder operators $\hat{b}, \hat{b}^\dagger$. Now we try to construct a coherent state $|\beta\rangle$ in this system composed of $|N-l, l\rangle\,, \hat{b}, \hat{b}^\dagger$ as coherent state $|\alpha\rangle$ in harmonic oscillator or of photon, but in approximate manner since we have a limitation on maximum number of quasi particle as follows

$$
\begin{aligned}
|\beta\rangle &= A_\beta \sum_{l=0}^{N} \frac{\beta^l}{\sqrt{l!}} |N-l, l\rangle \\
\hat{b}\,|\beta\rangle &= \beta A_\beta \left(\sum_{l=0}^{N} \frac{\beta^l}{\sqrt{l!}} |N-l, l\rangle - \frac{\beta^N}{\sqrt{N!}} |0, N\rangle \right) = \beta\,|\beta\rangle - \beta A_\beta \frac{\beta^N}{\sqrt{N!}} |0, N\rangle \,.
\end{aligned}
\tag{6.13}
$$

Now we see that $|\beta\rangle$ is approximately eigenstate of $\hat{b}$ except $\frac{\beta^N}{\sqrt{N!}}\,|0, N\rangle$. A_β can be determined from normalization condition $\langle\beta|\,\beta\rangle = 1$

$$
\begin{aligned}
\langle\beta|\,\beta\rangle &= |A_\beta|^2 \sum_{l=0}^{N} \frac{|\beta|^{2l}}{l!} = |A_\beta|^2 \, \exp(|\beta|^2) \frac{\Gamma(N+1, |\beta|^2)}{\Gamma(N+1)} = 1 \\
A_\beta &= \exp(-|\beta|^2/2) \sqrt{\frac{\Gamma(N+1)}{\Gamma(N+1, |\beta|^2)}}.
\end{aligned}
\tag{6.14}
$$

where $|\beta|^{2l}/l!$ is well-known Poisson distribution and $\Gamma(N+1, |\beta|)$ is upper incomplete gamma function

$$
\Gamma(s, x) = \int_x^\infty t^{s-1}\, e^{-t}\, \mathrm{d}t, \quad \Gamma(s, 0) = \Gamma(s). \tag{6.15}
$$

As $\Gamma(N+1, |\beta|^2) \to \Gamma(N+1)$, $|A_\beta| \to \exp(-|\beta|^2/2)$ and normalization factor A_β approaches to A_α of true coherent state $|\alpha\rangle$ in previous chapter. A deviation of $|A_\beta|^2$ from $|A_\alpha|^2$ when $|\alpha| = |\beta|$ can be written in terms of lower incomplete gamma function $\gamma(N+1, |\beta|)$ ($\gamma(s, x) + \Gamma(s, x) = \Gamma(s)$) as (note that $A_\beta > A_\alpha$ and α has equal value as β here)

$$
\frac{A_\beta^2 - A_\alpha^2}{A_\alpha^2} = \frac{\gamma(N+1, |\beta|^2)}{\Gamma(N+1)} \tag{6.16}
$$

As seen from Figure 6.1, $|\beta|^2/N \sim 0.8$ is enough for $A_\beta \simeq A_\alpha$ when $N = 100$. As N increases, higher $|\beta|^2/N$ value gives $A_\beta \simeq A_\alpha$ where $|\beta|^2$ gives mean particle number occupying mode 1 of $|\beta\rangle$.

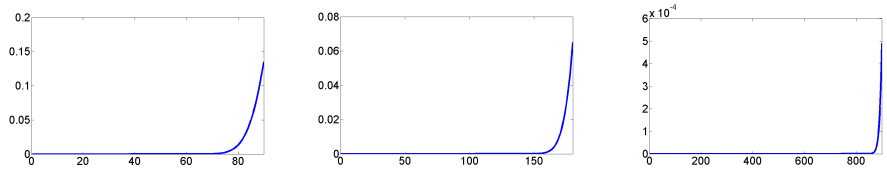

Figure 6.1: Plots of $\frac{\gamma(N+1,|\beta|)}{\Gamma(N+1)}$ for $N = 100, 200, 500$ (from left) where $|\beta|^2$ runs from 0 to $0.9N$.

We try to evaluate losses caused by repeated operation of $\hat{b}$ on $|\beta\rangle$ as

$$\beta^n |\beta\rangle - (\hat{b})^n |\beta\rangle = \beta^n A_\beta \left(\sum_{l=0}^{N} \frac{\beta^l}{\sqrt{l!}} |N-l, l\rangle - \sum_{l=N-n+1}^{N} \frac{\beta^l}{\sqrt{l!}} |N-l, l\rangle \right) \quad (6.17)$$

for n times operation of $\hat{b}$ on $|\beta\rangle$. It shows that as n increases $|\beta\rangle$ is less likely to be eigenstate of $\hat{b}$. And it is quantified in the right hand side of (6.17) that how much $(\hat{b})^n |\beta\rangle$ is different from $\beta^n |\beta\rangle$. Therefore following quantity

$$\begin{aligned} \frac{\sum_{l=N-n+1}^{N} \frac{|\beta|^{2l}}{l!}}{\sum_{l=0}^{N} \frac{|\beta|^{2l}}{l!}} &= \frac{\Gamma(N+1, |\beta|^2) - \Gamma(N+1)\frac{\Gamma(N-n+1,|\beta|^2)}{\Gamma(N-n+1)}}{\Gamma(N+1, |\beta|^2)} \\ &\simeq \exp(-|\beta|^2) \sum_{l=N-n+1}^{N} \frac{|\beta|^{2l}}{l!} \quad \text{for } |\beta|^2 < 0.8N \end{aligned} \quad (6.18)$$

quantifies how much $|\beta\rangle$ is away from coherent state for n times action of $\hat{b}$. As long as mean occupation number $\langle\beta| \hat{a}_1^\dagger \hat{a}_1 |\beta\rangle$ of mode 1 does not exceed $0.75N$ for $N \geq 100$, it is OK to think $|\beta\rangle$ as an eigenstate of quasi annihilation operator $\hat{b}$ for $N = 100$. And as N gets larger, it is more likely for approximate coherent state to behaves as eigenstate of $\hat{b}$ even with higher $|\beta|^2/N$.

Let $\beta = |\beta|e^{i\phi_\beta}$. Then $|\beta\rangle$ states, like phase state $|\phi, N, l_0\rangle$, consists over-complete basis of two mode state $|\Psi\rangle = \sum_l C_l |N-l, l\rangle$ through an integration over ϕ_β with fixed $|\beta|$ as follows

$$|\Psi\rangle = \frac{1}{A_\beta} \int_0^{2\pi} \frac{d\phi_\beta}{2\pi} \sum_l \frac{\sqrt{l!}}{|\beta|^l} C_l e^{-il\phi_\beta} |\beta\rangle \equiv \int_0^{2\pi} \frac{d\phi_\beta}{2\pi} C(\phi_\beta; |\beta|) |\beta\rangle \quad (6.19)$$

below we will abbreviate $C(\phi_\beta; |\beta|)$ to $C(\phi_\beta)$. A relation between C_l and $C(\phi_\beta)$ is given as

$$C(\phi_\beta) = \frac{1}{A_\beta} \sum_{l=0}^{N} \frac{\sqrt{l!}}{|\beta|^l} C_l e^{-il\phi_\beta}, \quad C_l = A_\beta \int_0^{2\pi} \frac{d\phi_\beta}{2\pi} \frac{|\beta|^l}{\sqrt{l!}} C(\phi_\beta) e^{il\phi_\beta}. \tag{6.20}$$

Also, it is stressed that property of constructed $\hat{b}, \hat{b}^\dagger$ and $|\beta\rangle$ is identical to those of 'truncated coherent state' and $\hat{a}, \hat{a}^\dagger$ of a harmonic oscillator following Glauber's definition [35] in finite-dimensional Hilbert space [38, 39].

Until now approximate coherent state is constructed where moving 1 particles from level 0 to level 1 is defined as a creation process. On the other hand, it is possible to construct approximate coherent state with letting moving 1 particles from level 1 to level 0 as creation process. Then it is possible to construct such $|\beta'\rangle$, $\hat{b}'$ and $\hat{b}'^\dagger$ as

$$\begin{aligned} \left|\beta'\right\rangle &= A_{\beta'} \sum_{l=0}^{N} \frac{\beta'^{N-l}}{\sqrt{(N-l)!}} \left|N-l, l\right\rangle, \ \hat{b}' = \lim_{\epsilon\to 0} \frac{1}{\sqrt{\hat{N}_1 + \epsilon}} \hat{a}_1^\dagger \hat{a}_0 \\ \hat{b}'^\dagger &= \lim_{\epsilon \to 0} \hat{a}_0^\dagger \hat{a}_1 \frac{1}{\sqrt{\hat{N}_1 + \epsilon}} \end{aligned} \tag{6.21}$$

where

$$A_{\beta'} = \exp(-|\beta'|^2/2) \sqrt{\frac{\Gamma(N+1)}{\Gamma(N+1, |\beta'|^2)}} \tag{6.22}$$

where ϵ is introduced to regularize singularity happens for $\hat{b'}^\dagger \left|N, 0\right\rangle$ without ϵ. Then acting two operators towards $|N-l, l\rangle$ gives

$$\begin{aligned} \hat{b}' \left|N-l, l\right\rangle &= \sqrt{N-l} \left|N-l+1, l-1\right\rangle \\ \hat{b'}^\dagger \left|N-l, l\right\rangle &= \sqrt{N-l+1} \left|N-l-, l+1\right\rangle \end{aligned} \tag{6.23}$$

as desired except $\hat{b'}^\dagger \left|N, 0\right\rangle = 0$ due to the fact that there is no particle left in mode 1 to be pulled down to mode 0. For a commutation relation $[\hat{b}', \hat{b'}^\dagger]$, a difference $1 - \langle\Psi| [\hat{b'}, \hat{b'}^\dagger] |\Psi\rangle$ occur by $|N, 0\rangle$ is

$$1 - \langle\Psi| [\hat{b'}, \hat{b'}^\dagger] |\Psi\rangle = \sum_{l=0}^{N} |C_l|^2 - \left(\sum_{l=1}^{N} |C_l|^2 - N|C_0|^2\right) = (N+1)|C_0|^2 \quad (6.24)$$

thus as long as $|C_0| \ll 1/\sqrt{N}$, it is always possible to construct bosonic ladder operator for arbitrary two-mode system. And repetitively acting $\hat{b'}^\dagger$ on $|\beta'\rangle$ gives

$$\begin{aligned} &\beta'^n |\beta'\rangle - (\hat{b})^n |\beta'\rangle \\ &= \beta'^n A_{\beta'} \left(\sum_{l=0}^{N} \frac{\beta'^{N-l}}{\sqrt{(N-l)!}} |N-l, l\rangle - \sum_{l=0}^{n-1} \frac{\beta'^{N-l}}{\sqrt{(N-l)!}} |N-l, l\rangle\right) \end{aligned} \quad (6.25)$$

$|\beta'\rangle$ becomes more exact coherent state for small $|\beta'|^2$, which corresponds to large $|\beta|^2$ value for $|\beta\rangle$ where $|\beta\rangle$ does not behave well as coherent state. Therefore with two types of approximate coherent states together we always have effective coherent state either $|\beta\rangle$ or $|\beta'\rangle$ depending on the value of $\langle \hat{a}_0^\dagger \hat{a}_0 \rangle : 0 \sim N$.

6.3 Relation between $|\beta\rangle$ and Phase State $|\phi, N, l_0\rangle$

In this section, similarities among $|\beta\rangle, |\beta'\rangle$ and $|\phi, N, l_0\rangle$ are investigated by comparing their C_l coefficient as two-mode state, SPDM and TPDM. In addition, transformation between them is discussed to further strengthen relation between phase state and approximate coherent state, which is key factor to express (or set up an analogy for) single-trap fragmented state as superposition of coherent states.

Let us first briefly rewrite a phase state $|\phi, N, l_0\rangle$ [22, 26]

$$\begin{aligned} |\phi, N, l_0\rangle &= \frac{(\hat{\psi}^\dagger_{\phi,N,l_0})^N}{\sqrt{N!}} |0\rangle = \frac{(\sqrt{N-l_0}\hat{a}_0^\dagger + e^{i\phi}\sqrt{l_0}\hat{a}_1^\dagger)^N}{\sqrt{N^N N!}} |0\rangle \\ &= \sum_{l=0}^{N} \sqrt{\frac{(N-l_0)^{N-l} l_0^l}{N^N N!}} e^{il\phi} \frac{N!}{(N-l)!l!} (\hat{a}_0^\dagger)^{N-l} (\hat{a}_1^\dagger)^l |0\rangle \\ &= \sum_{l=0}^{N} \frac{1}{\mathcal{N}_{N,l_0;l}} e^{il\phi} |N-l, l\rangle \end{aligned} \quad (6.26)$$

with

$$\mathcal{N}_{N,l_0;l} = \sqrt{\frac{N^N}{(N-l_0)^{N-l} l_0^l}} \sqrt{\frac{(N-l)!l!}{N!}} \tag{6.27}$$

And two-mode state $|\Psi\rangle = \sum_l C_l |N-l, l\rangle$ can be written as

$$\begin{aligned} |N-l,l\rangle &= \int_0^{2\pi} \frac{d\phi}{2\pi} \mathcal{N}_{N,l_0;l} e^{-il\phi} |\phi, N, l_0\rangle \\ |\Psi\rangle &= \int_0^{2\pi} \frac{d\phi}{2\pi} \sum_l \mathcal{N}_{N,l_0;l} C_l e^{-il\phi} |\phi, N, l_0\rangle \end{aligned} \tag{6.28}$$

by integration over ϕ with fixed l_0 value. It will be shown that $|\beta\rangle, |\beta'\rangle, |\phi, N, l_0\rangle$ gives the same SPDM (Single Particle Density Matrix) and TPDM (Two Particle Density Matrix) elements under certain conditions therefore they can be effectively considered as the same states. SPDM and TPDM completely determine first and second order correlation functions where two orbitals $\psi_0(\mathbf{r}), \psi_1(\mathbf{r})$ are given. First we show that by choosing $|\beta| = \sqrt{l_0}, |\beta'| = \sqrt{N-l_0}, \phi_\beta = \phi, \phi_{\beta'} = -\phi$ one gets most similar configurations of $|\beta\rangle$, $|\beta'\rangle$ and $|\phi, N, l_0\rangle$ to each other where $\beta = |\beta|e^{i\phi_\beta}$, $\beta' = |\beta'|e^{i\phi_{\beta'}}$. From (6.19) $|\phi, N, l_0\rangle$ can be written in terms of $|\beta\rangle$ as

$$\begin{aligned} |\phi, N, l_0\rangle &= \frac{1}{A_\beta} \int_0^{2\pi} \frac{d\phi_\beta}{2\pi} \sum_l \frac{e^{-il(\phi_\beta - \phi)}}{\mathcal{N}_{N,l_0;l}} \frac{\sqrt{l!}}{|\beta|^l} |\beta\rangle \\ &= \sqrt{\frac{N!}{N^N}} \frac{1}{A_\beta} \int_0^{2\pi} \frac{d\phi_\beta}{2\pi} \left(\sum_l \sqrt{\frac{(N-l_0)^{N-l} l_0^l}{(N-l)!|\beta|^l}} e^{-il(\phi_\beta - \phi)} \right) |\beta\rangle . \end{aligned} \tag{6.29}$$

Choosing $|\beta| = \sqrt{l_0}$ we get

$$\begin{aligned} |\phi, N, l_0\rangle &= \frac{e^{-il(\phi_\beta - \phi)}}{\mathcal{N}_{N,l_0;l}} \frac{\sqrt{l!}}{|\beta|^l} |\beta\rangle \\ &= \sqrt{\frac{N!}{N^N}} \frac{1}{A_\beta} \int_0^{2\pi} \frac{d\phi_\beta}{2\pi} \left(\sum_l \sqrt{\frac{(N-l_0)^{N-l}}{(N-l)!}} e^{-il(\phi_\beta - \phi)} \right) |\beta\rangle \end{aligned} \tag{6.30}$$

where a term in bracket becomes coefficient for $|\phi, N, l_0\rangle$ with $|\beta\rangle$ of different ϕ_β. $(N-l_0)^(N-l)/(N-l)!$ is proportional to Poisson distribution centered around $N-l_0$ and this can be well approximated by normal distribution of mean $N-l_0$ and variance $N-l_0$ for large $N-l_0$ (e.g. 10 20) with appropriate continuum approximation. Then

$$\sum_{l=0}^{N} \sqrt{\frac{(N-l_0)^{N-l}}{(N-l)!}} e^{-il(\phi_\beta - \phi)} \propto \sum_{l=0}^{N} e^{-\frac{(l-l_0)^2}{4(N-l_0)}} e^{-il(\phi_\beta - \phi)} \tag{6.31}$$

and it can be interpreted as discrete Fourier transform of normal distribution through summation over l except that there is no negative l and summation range is finite. This means that we get almost normal distribution of ϕ_β coefficient centered at $\phi_\beta = \phi$ and variance is $\sim \mathcal{O}(1/\sqrt{N-l_0})$ inverse proportional to $N-l_0$. Thus as l_0 gets smaller, we get more sharply peaked ϕ_β coefficient which leads to $|\phi, N, l_0\rangle \simeq |\beta\rangle$.

On the other hand, $|\phi, N, l_0\rangle$ in terms of $|\beta'\rangle$ with $|\beta'| = N - l_0$ is

$$|\phi, N, l_0\rangle = \sqrt{\frac{N!}{N^N}} \frac{1}{A_{\beta'}} \int_0^{2\pi} \frac{d\phi_{\beta'}}{2\pi} \left(e^{-iN\phi_{\beta'}} \sum_l \sqrt{\frac{l_0^l}{l!}} e^{il(\phi_{\beta'} + \phi)} \right) |\beta'\rangle \tag{6.32}$$

where a term in bracket becomes coefficient for $|\phi, N, l_0\rangle$ with $|\beta'\rangle$ of different $\phi_{\beta'}$. And it can be approximated as

$$e^{-iN\phi_{\beta'}} \sum_{l=0}^{N} \sqrt{\frac{l_0^l}{l!}} e^{il(\phi_{\beta'} + \phi)} \propto e^{-iN\phi_{\beta'}} \sum_{l=0}^{N} e^{-\frac{(l-l_0)^2}{4l_0}} e^{il(\phi_{\beta'} + \phi)} \tag{6.33}$$

and it can be interpreted as discrete Fourier transform of normal distribution through summation over l except that there is no negative l and summation range is finite. This means that we get almost normal distribution of $\phi_{\beta'}$ coefficient centered at $\phi_{\beta'} = -\phi$ and variance is $\sim \mathcal{O}(1/\sqrt{l_0})$ inverse proportional to l_0. Thus as $N - l_0$ gets smaller, we get more sharply peaked $\phi_{\beta'}$ coefficient which leads to $|\phi, N, l_0\rangle \simeq |\beta'\rangle$.

In summary, we expect that as l_0 gets smaller we get $|\phi, N, l_0\rangle \simeq |\beta\rangle$, and as l_0 gets larger we get $|\phi, N, l_0\rangle \simeq |\beta'\rangle$. And it will be confirmed in following figures below which compares C_l distribution of each $|\beta\rangle, |\beta'\rangle$ and $|\phi, N, l_0\rangle$. And $|\beta| = \sqrt{l_0}, |\beta'| = \sqrt{N - l_0}, \phi_\beta = \phi, \phi_{\beta'} = -\phi$ is optimal choice to compare three states $|\beta\rangle, |\beta'\rangle$ and $|\phi, N, l_0\rangle$.

And $|\beta\rangle$ and $|\beta'\rangle$ can be written in terms of $|\phi, N, l_0\rangle$ as

$$
\begin{aligned}
|\beta\rangle &= A_\beta \int_0^{2\pi} \frac{d\phi}{2\pi} \sum_{l=0}^{N} \mathcal{N}_{N,l_0;l} \frac{\beta^l}{\sqrt{l!}} e^{-il\phi} |\phi, N, l_0\rangle \\
|\beta'\rangle &= A_{\beta'} \int_0^{2\pi} \frac{d\phi}{2\pi} \sum_{l=0}^{N} \mathcal{N}_{N,l_0;l} \frac{\beta'^{N-l}}{\sqrt{(N-l)!}} e^{-il\phi} |\phi, N, l_0\rangle
\end{aligned} \tag{6.34}
$$

By letting $\beta = \sqrt{l_0} e^{i\phi_\beta}$ and $\beta' = \sqrt{N-l_0} e^{i\phi_{\beta'}}$,

$$
\begin{aligned}
|\beta\rangle &\propto \int_0^{2\pi} \frac{d\phi}{2\pi} \left(\sum_{l=0}^{N} \sqrt{\frac{(N-l)!}{(N-l_0)^{N-l}}} e^{-il(\phi-\phi_\beta)} \right) |\phi, N, l_0\rangle \\
|\beta'\rangle &\propto \int_0^{2\pi} \frac{d\phi}{2\pi} \left(\sum_{l=0}^{N} \sqrt{\frac{l!}{l_0^l}} e^{-il(\phi+\phi_{\beta'})} \right) |\phi, N, l_0\rangle
\end{aligned} \tag{6.35}
$$

thus it can be inferred that summation over l leaves contribution around $\phi \sim \phi_\beta$, $\phi \sim -\phi_{\beta'}$ and suppress else. In detail, factors in front of each $e^{-i(\phi-\phi_\beta)}$ and $e^{-i(\phi+\phi_{\beta'})}$, are square root inverse of Poisson distributions. And then we can find terms in brackets behaves as coefficients of each $|\beta\rangle$ and $|\beta'\rangle$ in representations with $|\phi, N, l_0\rangle$ basis, where coefficients in terms of ϕ are actually discrete Fourier transform (centered at each $\phi = \phi_\beta$ and $\phi = -\phi_{\beta'}$) of inverse square root of Poisson distributions. Unlike previous case, this time, this inverse square root of Poisson distribution almost diverges either at $l = 0$ or $l = N$. So a magnitude of ϕ coefficient very slowly varies and even ration between minimum value (at $\phi = \pi$) and maximum value (at $\phi = 0$) of magnitude of ϕ coefficient is only about 1.5 for $N = 100$ and $l_0 = 10$ when we try to express $|\beta\rangle$ ($|\beta|^2 = l_0$) in terms of $|\phi, N, l_0\rangle$ (and the same for $|\beta'\rangle$ while we consider to $|\beta'|^2 = N - l_0$). So we can express $|\phi, N, l_0\rangle$ directly in terms of $|\beta\rangle$ or $|\beta'\rangle$This strangeness comes from It is due to the fact that C_l distribution of $|\phi, N, l_0\rangle$ has narrower width than those of C_l distributions of $|\beta\rangle, |\beta'\rangle$, which is possible future outlook.

Now we compare $|C_l|^2$ distributions. Here we let $|C_l|^2(|\Psi\rangle)$ to be a $|C_l|^2$ distribution of two-mode state $|\Psi\rangle$.

$$
\begin{aligned}
|C_l|^2(|\beta\rangle) &= |A_\beta|^2 \frac{|\beta|^{2l}}{l!} = \frac{l_0^l}{l!} e^{-l_0} \frac{\Gamma(N+1)}{\Gamma(N+1, l_0)} \propto \frac{l_0^l}{l!} \\
|C_l|^2(|\beta'\rangle) &= |A_{\beta'}|^2 \frac{|\beta'|^{2(N-l)}}{(N-l)!} \propto \frac{(N-l_0)^{N-l}}{(N-l)!}
\end{aligned} \tag{6.36}
$$

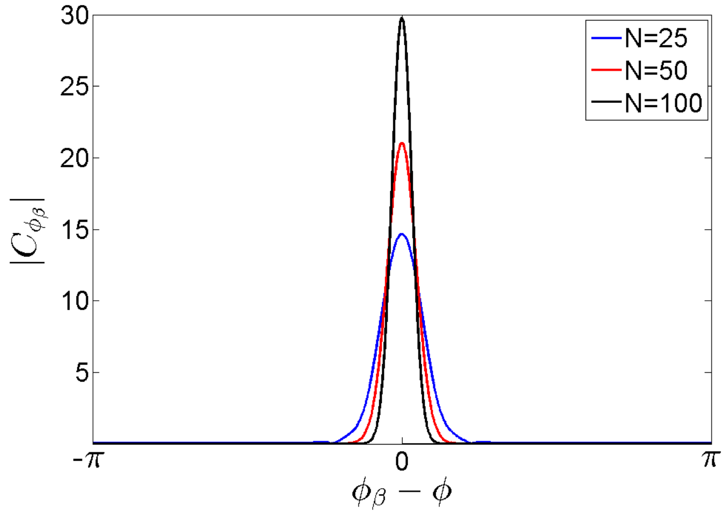

Figure 6.2: Plots of the $|C_{\phi_\beta}|$ distribution of $|\phi, N, l_0\rangle$, for $l_0 = |\beta|^2 = N/2$ with $N = 25, 50, 100$ [36].

and

$$|C_l|^2(|\phi, N, l_0\rangle) = \frac{N!}{N^N}\frac{(N-l_0)^{N-l}}{(N-l)!}\frac{l_0^l}{l!} \propto \frac{(N-l_0)^{N-l}}{(N-l)!}\frac{l_0^l}{l!} \tag{6.37}$$

We find that $|C_l|^2(|\phi, N, l_0\rangle)$ is proportional to multiplication of two Poisson distributions, $|C_l|^2(|\beta\rangle)$ and $|C_l|^2(|\beta'\rangle)$. Therefore $|C_l|^2(|\phi, N, l_0\rangle)$ always has narrower distribution width than $|C_l|^2(|\beta\rangle)$ and $|C_l|^2(|\beta'\rangle)$. We see that, as incomplete gamma functions above approaches to 1, all three $|C_l|^2$ distributions have maximum at $l = l_0$ and an expectation value of particles at level 1 is l_0. Furthermore, $|C_l|^2$ distributions of $|\beta\rangle$ and $|\beta'\rangle$ have property that they approaches to a normal distribution with small continuity correction on l for each $l_0 \geq 10$ and $N - l_0 \geq 10$ together with $\frac{\Gamma(N+1)}{\Gamma(N+1,l_0)}, \frac{\Gamma(N+1)}{\Gamma(N+1,N-l_0)} \simeq 1$ [ref]. And $|C_l|^2$ distribution for the phase state $|\phi, N, l_0\rangle$ is expected to be also a normal distribution with the same mean value but smaller standard deviation since $\exp(-x^2/2\sigma_1^2)\exp(-x^2/2\sigma_2^2) = \exp(-x^2/\sigma_3^2)$ where $\sigma_3 = \sqrt{\sigma_1^2\sigma_2^2/(\sigma_1^2+\sigma_2^2)} < \sigma_1, \sigma_2$. This indirectly shows that as l_0 approaches near either to 0 or N, one of σ_1 or σ_2 increases a lot and $\sigma_3 \simeq \sigma_1$ or $\sigma_3 \simeq \sigma_2$. Then we get $|\beta\rangle \simeq |\phi, N, l_0\rangle$ or $|\beta'\rangle \simeq |\phi, N, l_0\rangle$. And it is shown in following figures. Figures are plots of three $|C_l|^2$ distributions for $l_0 = 0.1N, 0.2N, 0.3N, 0.4N$ (from left) and $N = 100$ (By looking at $|\beta'\rangle$ for $l_0 = 0.1N, 0.2N, 0.3N, 0.4N$ we can get a result of $|\beta\rangle$ for

$l_0 = 0.9N, 0.8N, 0.7N, 0.6N$ and vice versa.) As l_0 approaches to $0.5N$, one can

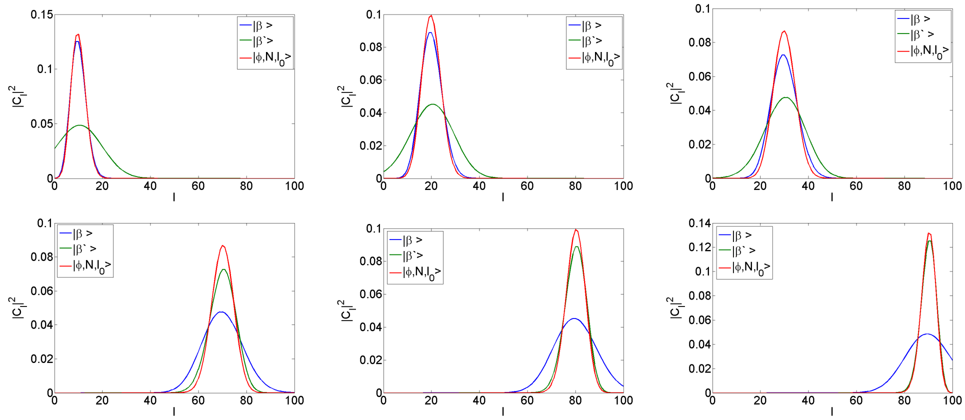

Figure 6.3: Plots of the $|C_l|^2$ distribution of $|\phi, N, l_0\rangle$ (red), $|\beta\rangle$ (blue), $|\beta'\rangle$ (green) for with $N = 100$ for different $l_0 = |\beta|^2 = N - |\beta'|^2 = 0.1N, 0.2N, 0.3N$ (Top from left) and $l_0 = |\beta|^2 = N - |\beta'|^2 = 0.7N, 0.8N, 0.9N$ (Bottom from left) [36].

see that three $|C_l|^2$ distributions coincide to each other except slightly narrower distribution for the phase state $|\phi, N, l_0\rangle$. SPDM $\rho_{ij} \equiv \langle \hat{a}_i^\dagger \hat{a}_j \rangle$ and TPDM elements $\rho_{ijkl} \equiv \langle \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l \rangle$ for three states $|\beta\rangle$, $|\beta'\rangle$ and $|\phi, N, l_0\rangle$ varying l_0 and N. First we conduct calculations for $\rho_{00} = N - \rho_{11}$, ρ_{0000}, ρ_{1111}, $\rho_{0110} = \rho_{1010} = \rho_{0101} = \rho_{1001}$ which depend only on $|C_l|^2$ as follows

$$\begin{aligned} \rho_{00} &= \sum_{l=0}^{N-1} (N-l)|C_l|^2, \ \rho_{0000} = \sum_{l=0}^{N-2} (N-l)(N-l-1)|C_l|^2, \\ \rho_{1111} &= \sum_{l=2}^{N} l(l-1)|C_l|^2, \ \rho_{0110} = \sum_{l=1}^{N-1} (N-l)l|C_l|^2. \end{aligned} \tag{6.38}$$

And we get Table 6.1 (We can get a result of $|\beta\rangle$ for $l_0 = 0.9N, 0.8N$ from $|\beta'\rangle$ for $l_0 = 0.1N, 0.2N$ and vice versa),

$\rho_{01} = \rho_{10}^*$, $\rho_{0001} = \rho_{0010} = \rho_{0100}^* = \rho_{1000}^*$, $\rho_{0111} = \rho_{1011} = \rho_{1101}^* = \rho_{1110}^*$, $\rho_{0011} = \rho_{1100}^*$ which depend on the phase of C_l as well as $|C_l|^2$, can

$N=100,\ l_0=0.1N$	$\lvert\beta\rangle$	$\lvert\beta'\rangle$	$\lvert\phi,N,l_0\rangle$
ρ_{00}/N	0.900	0.876	0.900
ρ_{0000}/N^2	0.802	0.764	0.802
ρ_{1111}/N^2	0.01	0.0199	0.0099
ρ_{0110}/N^2	0.0890	0.1031	0.0891
$N=100,\ l_0=0.2N$	$\lvert\beta\rangle$	$\lvert\beta'\rangle$	$\lvert\phi,N,l_0\rangle$
ρ_{00}/N	0.800	0.797	0.800
ρ_{0000}/N^2	0.634	0.634	0.634
ρ_{1111}/N^2	0.04	0.0466	0.0396
ρ_{0110}/N^2	0.1580	0.1546	0.1584

Table 6.1: Comparison of SPDM and TPDM among $|\beta\rangle, |\beta'\rangle, |\phi, N, l_0\rangle$

be calculated from

$$\begin{aligned}
\rho_{01} &= \sum_{l=1}^{N} C_{l-1}^* C_l \sqrt{(N-l+1)l} \\
\rho_{0001} &= \sum_{l=1}^{N-1} C_{l-1}^* C_l (N-l)\sqrt{(N-l+1)l}, \\
\rho_{0111} &= \sum_{l=2}^{N} C_{l-1}^* C_l (l-1)\sqrt{(N-l+1)l} \\
\rho_{0011} &= \sum_{l=2}^{N} C_{l-2}^* C_l \sqrt{(N-l+1)(N-l+2)l(l-1)}.
\end{aligned} \tag{6.39}$$

Here $\phi = \pi/4, \pi/2$ cases are calculated in tables 6.2,6.3. One can see that magnitude of SPDM is well described with l_0. Magnitude of TPDM also is, for example, $|\rho_{0011}| \simeq |\rho_{0110}| \simeq (N-l_0)l_0$. Phases of both SPDM and TPDM are both in an excellent agreement with given ϕ, i.e. phase from $\hat{a}_0^\dagger \hat{a}_1$ is exactly ϕ. Deviation of TPDM from mean value was slightly larger than that of SPDM since TPDM is more sensitive to distribution of C_l. Three states are expected to yield also similar values for higher order elements. For example, if we think of three particle density matrix, those elements are just expectation values for variables of third power, e.g. $(N-l)^3, l^3, (N-l)^2 l, \cdots$, from simple Poisson (or Normal) distribution of l ($|\beta\rangle$ and $|\phi, N, l_0\rangle$ for $l_0 \leq N$) or $N-l$ ($|\beta'\rangle$ and $|\phi, N, l_0\rangle$ for $l_0 \geq N$).

$N=100, l_0=0.1N, \phi=\frac{\pi}{4}$	$\vert\beta\rangle$	$\vert\beta'\rangle$	$\vert\phi,N,l_0\rangle$
$\vert\rho_{01}\vert/N$	0.300	0.314	0.300
$\vert\rho_{0001}\vert/N^2$	0.267	0.267	0.267
$\vert\rho_{0111}\vert/N^2$	0.0298	0.0443	0.0297
$\vert\rho_{0011}\vert/N^2$	0.0895	0.1072	0.0891
$\arg(\rho_{01})$	0.250π	0.250π	0.250π
$\arg(\rho_{0001})$	0.250π	0.250π	0.250π
$\arg(\rho_{0111})$	0.250π	0.250π	0.250π
$\arg(\rho_{0011})$	0.500π	0.500π	0.500π

$N=100, l_0=0.1N, \phi=\frac{\pi}{2}$	$\vert\beta\rangle$	$\vert\beta'\rangle$	$\vert\phi,N,l_0\rangle$
$\vert\rho_{01}\vert/N$	0.300	0.314	0.300
$\vert\rho_{0001}\vert/N^2$	0.267	0.267	0.267
$\vert\rho_{0111}\vert/N^2$	0.0298	0.0443	0.0297
$\vert\rho_{0011}\vert/N^2$	0.0895	0.1072	0.0891
$\arg(\rho_{01})$	0.500π	0.500π	0.500π
$\arg(\rho_{0001})$	0.500π	0.500π	0.500π
$\arg(\rho_{0111})$	0.500π	0.500π	0.500π
$\arg(\rho_{0011})$	π	π	π

Table 6.2: For $\phi=\frac{\pi}{4}$ (top) and $\phi=\frac{\pi}{2}$ (bottom) with $N=100, l_0=0.1N$.

$N=100, l_0=0.2N, \phi=\frac{\pi}{4}$	$\vert\beta\rangle$	$\vert\beta'\rangle$	$\vert\phi,N,l_0\rangle$
$\vert\rho_{01}\vert/N$	0.400	0.392	0.400
$\vert\rho_{0001}\vert/N^2$	0.316	0.305	0.317
$\vert\rho_{0111}\vert/N^2$	0.0795	0.0836	0.0792
$\vert\rho_{0011}\vert/N^2$	0.159	0.159	0.158
$\arg(\rho_{01})$	0.250π	0.250π	0.250π
$\arg(\rho_{0001})$	0.250π	0.250π	0.250π
$\arg(\rho_{0111})$	0.250π	0.250π	0.250π
$\arg(\rho_{0011})$	0.500π	0.500π	0.500π

$N=100, l_0=0.2N, \phi=\frac{\pi}{2}$	$\vert\beta\rangle$	$\vert\beta'\rangle$	$\vert\phi,N,l_0\rangle$
$\vert\rho_{01}\vert/N$	0.400	0.392	0.400
$\vert\rho_{0001}\vert/N^2$	0.316	0.305	0.317
$\vert\rho_{0111}\vert/N^2$	0.0795	0.0836	0.0792
$\vert\rho_{0011}\vert/N^2$	0.159	0.159	0.158
$\arg(\rho_{01})$	0.500π	0.500π	0.500π
$\arg(\rho_{0001})$	0.500π	0.500π	0.500π
$\arg(\rho_{0111})$	0.500π	0.500π	0.500π
$\arg(\rho_{0011})$	π	π	π

Table 6.3: For $\phi=\frac{\pi}{4}$ (top) and $\phi=\frac{\pi}{2}$ (bottom) with $N=100, l_0=0.2N$.

Chapter 7

An Analogy to Photonic Schrödinger Cat State

In previous chapter, $|\beta\rangle$ was confirmed as well behaving (approximate) coherent state and as almost the same quantum many-body state as phase state $|\phi, N, l_0\rangle$ effectively for $|\beta|^2 = l_0 \leq N/2$ with N larger than few tens e.g. 50 ($|\beta|^2 = l_0 > N/2$ there exists $|\beta'\rangle$). From discussions in 5.2.1 and 6.3, it is inferred that single-trap fragmented state, pure state, can be expressed as superposition of $|\beta\rangle$ and $|-\beta\rangle$ of $\phi_\beta = \pi/2$ where $|\beta\rangle \simeq |\phi, N, l_0\rangle$ with $|\beta|^2 = l_0$, $\phi = \phi_\beta$ (for $l_0 \leq N/2$ and $|\beta'\rangle$ used for $l_0 \geq N/2$).

In this chapter, we first try direct transformation of single-trap fragmented state into $|\beta\rangle$ basis by considering two-mode state with Gaussian $|C_l|$ distribution. Gaussian $|C_l|$ distribution can describe Fock state (number state) $|N - l, l\rangle$, BEC, and NPC state. It is shown that pursuing explicit equality does not work beyond certain limitation due to subtle mathematical issue, but together with previous observations, still it is possible to express single-trap fragmented state as cat state, or superposition of coherent states. Thus relation between coefficients of $|\text{Even}\rangle, |\text{Odd}\rangle$ superposition and $|\beta\rangle, |-\beta\rangle$ superposition are investigated to identify fragmented state as cat state. Furthermore, relation between fluctuation in density-density correlation and quadrature fluctuation is studied. This relation enables detection of quadrature fluctuation of $\hat{b}$ from measurement of density and density-density correlation.

7.1 $C(\phi_\beta)$ for Gaussian $|C_l|$ Distribution

To find what we can get from two-mod state $|\Psi\rangle$ written in terms of $|\beta\rangle$ basis, we apply following ansatz for C_l of

$$|\Psi\rangle = \sum_{l=0}^{N} C_l \, |N-l, l\rangle = \sum_{l=0}^{N} C_l \frac{(\hat{a}_0^\dagger)^{N-l}(\hat{a}_1^\dagger)^l}{\sqrt{(N-l)!l!}} \, |0\rangle \tag{7.1}$$

with Gaussian $|C_l|^2$ distribution of mean $l = l_0$ and variance σ^2 as follows

$$C_l = \frac{1}{(2\pi\sigma^2)^{1/4}} e^{-\frac{(l-l_0)^2}{4\sigma^2}} e^{i\phi_l}, \quad |C_l|^2 = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(l-l_0)^2}{2\sigma^2}} \tag{7.2}$$

where $C_l = |C_l| e^{i\phi_l}$. Then an expectation values of number operators $\hat{N}_0 = \hat{a}_0^\dagger \hat{a}_0, \ \hat{N}_1 = \hat{a}_1^\dagger \hat{a}_1$ are determined from $|C_l|^2$ as

$$\begin{aligned} &\langle\Psi| \hat{N}_0 |\Psi\rangle = \sum_{l=0}^{N} (N-l)|C_l|^2, \ \ \langle\Psi| \hat{N}_1 |\Psi\rangle = \sum_{l=0}^{N} l|C_l|^2 \\ &\langle\Psi| (\hat{N}_0)^m (\hat{N}_1)^n |\Psi\rangle = \sum_{l=0}^{N} (N-l)^m \, l^n |C_l|^2 \end{aligned} \tag{7.3}$$

where m, n are 0 or positive integer. This type of C_l distribution is able to express wanted $\langle\Psi| \hat{N}_0 |\Psi\rangle, \langle\Psi| \hat{N}_1 |\Psi\rangle$ with controlled fluctuation with value of σ. These are nothing but random variables ($\hat{N}_0 \to N-l, \ \hat{N}_1 \to l$) evaluated over *truncated* normal probabilistic distribution $|C_l|^2$ of mean $l = l_0$ and variance σ^2 as long as contribution from missing $l < 0, \ l > N$ is not too large.

With this formalism we can deal with following 3 types of states

- Fock state $|N-l, l\rangle$
- BEC state $|\text{BEC}\rangle = \frac{(\hat{a}_{BEC}^\dagger)^N}{\sqrt{N!}} \, |0\rangle$
- Negative Pair Coherent (NPC) state $|NPC\rangle$ dealt in [8, 21]

And each are

- Fock state $|N-l, l\rangle$ can be achieved by $l_0 = l$ and $\sigma \to 0$. For discrete l, $\sigma \to 0$ corresponds to $C_l = \delta_{l,l_0}$.

- With respect to $|\text{BEC}\rangle$, for any $\hat{a}^\dagger_{BEC}$ there exist $\hat{a}^\dagger_0, \hat{a}^\dagger_1$ such that $\hat{a}^\dagger_{BEC} = (\sqrt{N-l_0}\hat{a}^\dagger_0 + \sqrt{l_0}\hat{a}^\dagger_1)/\sqrt{N}$. Then we have

$$
\begin{aligned}
|\text{BEC}\rangle &= \frac{(\sqrt{N-l_0}\hat{a}^\dagger_0 + \sqrt{l_0}\hat{a}^\dagger_1)^N}{\sqrt{N!N^N}} |0\rangle \\
&\propto \sum_{l=0}^{N} \sqrt{\frac{(N-l_0)^{N-l}}{(N-l)!}} \sqrt{\frac{l_0^l}{l!}} \, |N-l, l\rangle \\
&\propto \sum_{l=0}^{N} \exp\left(-\frac{(l-l_0)^2}{4}\left(\frac{1}{N-l_0} + \frac{1}{l_0}\right)\right)
\end{aligned}
\tag{7.4}
$$

  for large enough $l_0, N-l_0$ to apply normal distribution approximation to Poisson distributions $(N-l_0)^{N-l}/(N-l)!$ and $l_0^l/l!$. Therefore in principle $|\text{BEC}\rangle$ can be described with Gaussian $|C_l|$ distribution of mean at $l = l_0$ and variance $(N-l_0)l_0/N$.

- NPC state $|NPC\rangle$, discussed in 3.2.1, has $|C_l|$ distribution which is Gaussian with variance $\mathcal{O}(N)$ and sign change $\text{sgn}(C_l C_{l+2}) = -1$ under continuum limit applied to discrete l [8, 19, 21].

From (6.20) we find $C(\phi_\beta)$ for C_l in (7.2). Then

$$
C(\phi_\beta) = \frac{1}{A_\beta}\sum_{l=0}^{N}\frac{\sqrt{l!}}{|\beta|^l}C_l e^{-il\phi_\beta} = \frac{1}{A_\beta}\frac{1}{(2\pi\sigma^2)^{1/4}}\sum_{l=0}^{N}\frac{\sqrt{l!}}{|\beta|^l}e^{-\frac{(l-l_0)^2}{4\sigma^2}}e^{i(\phi_l - l\phi_\beta)} \tag{7.5}
$$

And since

$$
|\beta\rangle = A_\beta \sum_{l=0}^{N}\frac{|\beta|^l}{\sqrt{l!}}e^{il\phi_\beta}\, |N-l, l\rangle \tag{7.6}
$$

has C_l distribution maximum at $l = |\beta|^2$, so it is natural to ‘ choose’ the value of $|\beta|$ to be $|\beta|^2 = l_0$ to describe $|C_l| \propto e^{-(l-l_0)/4\sigma^2}$ centered at $l = l_0$. Then (7.5) reduces to

$$
\begin{aligned}
C(\phi_\beta) &\propto \sum_{l=0}^{N} e^{-\frac{(l-|\beta|^2)^2}{4|\beta|^2}} e^{-\frac{(l-l_0)^2}{4\sigma^2}} e^{i(\phi_l - l\phi_\beta)} \\
&= \sum_{l=0}^{N} \exp\left(-\frac{(l-|\beta|^2)^2}{4}\left(\frac{1}{\sigma^2} - \frac{1}{|\beta|^2}\right)\right) e^{i(\phi_l - l\phi_\beta)}
\end{aligned}
\tag{7.7}
$$

with chosen $|\beta|^2 = l_0$ and normal distribution approximation applied to $\sqrt{l!}/|\beta|^l$ which is an inverse square root of Poisson distribution $|\beta|^{2l}/l!$. From (7.7), one can see that $C(\phi_\beta)$ is proportional to

$$\begin{cases} \sum_{l=0}^{N} \exp\left(-\frac{(l-|\beta|^2)^2}{4}\left|\frac{1}{\sigma^2}-\frac{1}{|\beta|^2}\right|\right) e^{i(\phi_l - l\phi_\beta)} & \text{if } \sigma^2 < |\beta|^2 = l_0 \\ \sum_{l=0}^{N} \exp\left(\frac{(l-|\beta|^2)^2}{4}\left|\frac{1}{\sigma^2}-\frac{1}{|\beta|^2}\right|\right) e^{i(\phi_l - l\phi_\beta)} & \text{if } \sigma^2 > |\beta|^2 = l_0. \end{cases} \tag{7.8}$$

And if $\phi_l = \phi_{l-1} + \Delta\phi = l\Delta\phi + \phi_0$ we choose $\phi_0 = 0$, then for $\sigma < |\beta|^2$ we have $|C(\phi_\beta)|$ proportional to

$$\left|\sum_{l=0}^{N} \exp\left(-\frac{(l-|\beta|^2)^2}{4}\left|\frac{1}{\sigma^2}-\frac{1}{|\beta|^2}\right|\right) e^{il(\Delta\phi - \phi_\beta)}\right| \propto \exp\left(-\frac{(\phi_\beta - \Delta\phi)^2}{\left|\frac{1}{\sigma^2}-\frac{1}{|\beta|^2}\right|}\right) \tag{7.9}$$

near $\phi_\beta = \Delta\phi$. We would like note that $\Delta\phi = 0$ corresponds to constant ϕ_l. Therefore when $\sigma^2 < |\beta|^2$ it is expected that Gaussian (or similar to Gaussian) C_l distribution with phase $\phi_l = l\Delta\phi$ (up to constant) yields well-peaked $C(\phi_\beta)$ distribution around $\phi_\beta = \Delta\phi$. To see how this $|\beta\rangle$ states is useful, we are going to show an example for NPC state of $\sigma^2 = 0.5|\beta|^2 < |\beta|^2$.

7.2 Identifying Single-Trap Fragmented State as a "Photonic" Cat State

$\mathrm{sgn}(C_l, C_{l\pm 2}) = -1$ is not enough to determine phase ϕ_l of C_l, but here we will consider $C_l \in \mathbb{R}$ and positive C_0, C_1. Then we first assume

$$\phi_0 = 0, \phi_1 = 0, \phi_2 = \pi, \phi_3 = \pi, \phi_4 = 0, \cdots, \quad \phi_l = \phi_{l+4}. \tag{7.10}$$

Now for such NPC state with Gaussian $|C_l|$ distribution of $\sigma^2 = 0.5|\beta|^2$, we have following $C(\phi_\beta)$ for $N = 100, 200, 400$ (from left) with the same $l_0 = 0.1N$ and $|\beta|^2 = l_0$. (note: in the plot, $\phi_\beta : -\pi \sim \pi$ was used instead of $0 \sim 2\pi$)

As N gets larger, we have narrower $|C(\phi_\beta)|$ distribution around $\phi_\beta = \pi/2$ and $\phi_\beta = -\pi/2$ $(= 3\pi/2)$. Further, irrelevant of a value of $\phi_0 - \phi_1$, $\mathrm{sgn}(C_l, C_{l\pm 2}) = -1$ enables us to write down as

$$|\Psi\rangle \simeq C_\beta |\beta\rangle + C_{-\beta} |-\beta\rangle, \quad \beta = |\beta| e^{i\frac{\pi}{2}},\ |\beta|^2 = l_0 \tag{7.11}$$

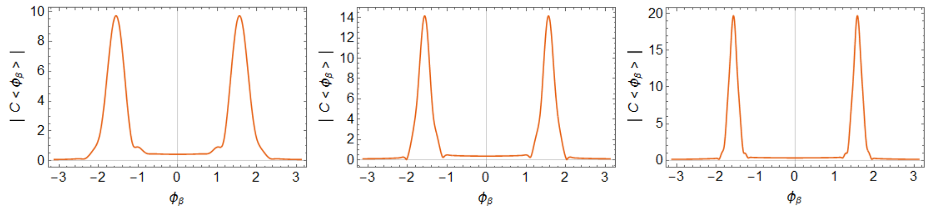

Figure 7.1: Plots of $|C(\phi_\beta)|$ for $N = 100, 200, 400$ (from left) with the same $l_0 = 0.1N$ and $|\beta|^2 = l_0$.

where $\beta = |\beta|e^{\frac{\pi}{2}i}$. From here, we fix β as $\beta = |\beta|e^{\frac{\pi}{2}i}$ to the end of this subsection. Then we have from (6.13)

$$|\Psi\rangle \simeq C_\beta |\beta\rangle + C_{-\beta} |-\beta\rangle = A_\beta \sum_{l=0}^{N} \frac{|\beta|^l}{\sqrt{l!}} \left(C_\beta e^{i\frac{\pi}{2}l} + C_{-\beta} e^{-i\frac{\pi}{2}l} \right) |N-l, l\rangle \quad (7.12)$$

and

$$\begin{aligned} \phi_l &= \text{Arg}\left(C_\beta i^l + C_{-\beta}(-i)^l \right) = \text{Arg}\left((C_\beta + (-1)^l C_{-\beta})\, i^l \right) \\ &= \text{Arg}\left(C_\beta + C_{-\beta}(-1)^l \right) + \frac{l\pi}{2} \end{aligned} \quad (7.13)$$

With ϕ_l given in (7.10), up to small error it is possible to write $|\Psi\rangle$ as

$$|\Psi\rangle \simeq \frac{1-i}{2} |\beta\rangle + \frac{1+i}{2} |-\beta\rangle, \quad \beta = |\beta|e^{i\frac{\pi}{2}}, \ |\beta|^2 = l_0 \quad (7.14)$$

which is *a superposition of two approximate coherent states* $|\beta\rangle$ and $|-\beta\rangle$. And these values of C_β and $C_{-\beta}$ gives

$$\phi_l = \text{Arg}\left(C_\beta + (-1)^l C_{-\beta} \right) + \frac{l\pi}{2} = \text{Arg}\left(\frac{1-i}{2} + \frac{1+i}{2}(-1)^l \right) + \frac{l\pi}{2} \quad (7.15)$$

with $\phi_0, \phi_1, \phi_2, \phi_3$ as

$$
\begin{aligned}
\phi_0 &= \mathrm{Arg}\left(\frac{1-i}{2} + \frac{1+i}{2}\right) = 0 \\
\phi_1 &= \mathrm{Arg}\left(\frac{1-i}{2} - \frac{1+i}{2}\right) + \frac{\pi}{2} = \mathrm{Arg}(-i) + \frac{\pi}{2} = 0 \\
\phi_2 &= \mathrm{Arg}\left(\frac{1-i}{2} + \frac{1+i}{2}\right) + \pi = \pi \\
\phi_3 &= \mathrm{Arg}\left(\frac{1-i}{2} - \frac{1+i}{2}\right) + \frac{3\pi}{2} = \mathrm{Arg}(-i) + \frac{3\pi}{2} = 0
\end{aligned}
\tag{7.16}
$$

and

$$
\phi_{l+4} = \mathrm{Arg}\left(\frac{1-i}{2} + \frac{1+i}{2}(-1)^l\right) + \frac{\pi}{2} = \phi_l \tag{7.17}
$$

which agrees with (7.10). Therefore two peak structures in Figure 7.1 matches with (7.9). Also we observe that with $\langle -\beta|\ \beta\rangle = A_\beta^2 \sum_{l=0}^{N} \frac{(-|\beta|^2)^l}{l!} \simeq e^{-2|\beta|^2}$

$$
\begin{aligned}
|\mathrm{Even}\rangle &\equiv \frac{1}{\sqrt{2(1+e^{-2|\beta|^2})}} \left(|\beta\rangle + |-\beta\rangle\right) \\
&= \frac{A_\beta}{\sqrt{2(1+e^{-2|\beta|^2})}} \sum_{l=0}^{N} \frac{|\beta|^l}{\sqrt{l!}} \left(e^{i\frac{\pi}{2}l} + e^{-i\frac{\pi}{2}l}\right) |N-l, l\rangle \\
&= \frac{A_\beta}{\sqrt{2(1+e^{-2|\beta|^2})}} \sum_{l=0}^{N} \frac{|\beta|^l}{\sqrt{l!}} \left(i^l(1+(-1)^l)\right) |N-l, l\rangle
\end{aligned}
\tag{7.18}
$$

where $(1+(-1)^l)$ yields $C_l = 0$ for odd l which means $\frac{1}{\sqrt{2}}\left(|\beta\rangle + |-\beta\rangle\right)$ has only $|N-l, l\rangle$ with even l. Assuming $e^{-2|\beta|^2} \simeq 0$, we have

$$
|\mathrm{Even}\rangle = \frac{1}{\sqrt{2}}(|\beta\rangle + |-\beta\rangle) \tag{7.19}
$$

And

$$
\begin{aligned}
|\mathrm{Odd}\rangle &\equiv \frac{1}{i\sqrt{2(1-e^{-2|\beta|^2})}} \left(|\beta\rangle - |-\beta\rangle\right) \\
&= \frac{A_\beta}{i\sqrt{2(1-e^{-2|\beta|^2})}} \sum_{l=0}^{N} \frac{|\beta|^l}{\sqrt{l!}} \left(e^{i\frac{\pi}{2}l} - e^{-i\frac{\pi}{2}l}\right) |N-l, l\rangle \\
&= \frac{A_\beta}{i\sqrt{2(1-e^{-2|\beta|^2})}} \sum_{l=0}^{N} \frac{|\beta|^l}{\sqrt{l!}} \left(i^l(1-(-1)^l)\right) |N-l, l\rangle
\end{aligned}
\tag{7.20}
$$

where $(1-(-1)^l)$ yields $C_l = 0$ for even l which means $\frac{1}{\sqrt{2}}\left(|\beta\rangle + |-\beta\rangle\right)$ has only

$|N-l,l\rangle$ with odd l. Assuming $e^{-2|\beta|^2} \simeq 0$, we have

$$|\text{Odd}\rangle = \frac{1}{i\sqrt{2}}(|\beta\rangle - |-\beta\rangle) \tag{7.21}$$

Now we are going to find a relation between $|\beta\rangle\,,\ |-\beta\rangle$ and $|\text{Even}\rangle\,,\ |\text{Odd}\rangle$. Therefore we first find a ratio $r = |C_\beta|/|C_{-\beta}|$ and $\theta = \text{Arg}(C_\beta/C_{-\beta})$ of

$$C_\beta\,|\beta\rangle + C_{-\beta}\,|-\beta\rangle = C_\beta\left(|\beta\rangle + re^{i\theta}\,|-\beta\rangle\right) \tag{7.22}$$

with normalization condition

$$\begin{aligned} &1 = |C_\beta|^2\left(1 + r^2 + 2r\cos\theta\,\langle-\beta|\,\beta\rangle\right) \to |C_\beta| = \frac{1}{\sqrt{1 + r^2 + 2r\cos\theta e^{-2|\beta|^2}}} \\ &\text{where } \langle-\beta|\,\beta\rangle = A_\beta^2\sum_{l=0}^{N}\frac{(-|\beta|^2)^l}{l!} \simeq e^{-2|\beta|^2} \end{aligned} \tag{7.23}$$

for general NPC state [21]

$$|NPC\rangle \simeq c\left(|\text{Even}\rangle + ue^{i\theta_k}\,|\text{Odd}\rangle\right) \tag{7.24}$$

with normalization condition

$$1 = |c|^2\left(1 + u^2\right), \quad |c| = \frac{1}{\sqrt{1+u^2}} \tag{7.25}$$

where u and θ_k each denotes for magnitude ratio and relative phase between even l sector and odd l sector. Here we think simple case, $|\text{Even}\rangle$ and $|\text{Odd}\rangle$ in (7.19) and (7.21) with $e^{-2|\beta|^2} \simeq 0$. Then

$$\begin{aligned} &a\left(|\text{Even}\rangle + ue^{i\theta_k}\,|\text{Odd}\rangle\right) = c\left(\frac{1}{\sqrt{2}}\,(|\beta\rangle + |-\beta\rangle) + \frac{ue^{i\theta_k}}{\sqrt{2}i}\,(|\beta\rangle - |-\beta\rangle)\right) \\ &= \frac{c}{\sqrt{2}}\left(1 - ue^{i(\theta_k+\pi/2)}\right)\left(|\beta\rangle + \frac{1 + ue^{i(\theta_k+\pi/2)}}{1 - ue^{i(\theta_k+\pi/2)}}\,|-\beta\rangle\right) \\ &= C_\beta\left(|\beta\rangle + \frac{1 + ue^{i(\theta_k+\pi/2)}}{1 - ue^{i(\theta_k+\pi/2)}}\,|-\beta\rangle\right) \end{aligned} \tag{7.26}$$

and

$$r = \left| \frac{1 + ue^{i(\theta_k + \pi/2)}}{1 - ue^{i(\theta_k + \pi/2)}} \right|, \quad \theta = \mathrm{Arg}\left(\frac{1 + ue^{i(\theta_k + \pi/2)}}{1 - ue^{i(\theta_k + \pi/2)}} \right). \tag{7.27}$$

We put temporarily $\vartheta = \theta_k + \pi/2$, and

$$\begin{aligned}
\frac{1 + ue^{i\vartheta}}{1 - ue^{i\vartheta}} &= \frac{1 + u\cos\vartheta + iu\sin\vartheta}{1 - u\cos\vartheta - iu\sin\vartheta} \\
&= \frac{(1 + u\cos\vartheta + iu\sin\vartheta)(1 - u\cos\vartheta + iu\sin\vartheta)}{(1 - u\cos\vartheta)^2 + u^2\sin^2\vartheta} \\
&= \frac{(1 + iu\sin\vartheta)^2 - u^2\cos^2\vartheta}{(1 - u\cos\vartheta)^2 + u^2\sin^2\vartheta} = \frac{(1 - u^2) + 2iu\sin\vartheta}{(1 + u^2) - 2u\cos\vartheta}.
\end{aligned} \tag{7.28}$$

Finally from $\sin(\theta_k + \pi/2) = \cos\theta_k$ and $\cos(\theta_k + \pi/2) = -\sin\theta_k$ we get

$$r = \left| \frac{(1 - u^2) + 2iu\cos\theta_k}{(1 + u^2) + 2u\sin\theta_k} \right| \tag{7.29}$$

and

$$\theta = \mathrm{Arg}\left((1 - u^2) + 2iu\cos\theta_k\right) = \tan^{-1}\left(\frac{2u\cos\theta_k}{1 - u^2}\right) \tag{7.30}$$

Here we find that both $u = 0$ and $u \to \infty$ leads to $\theta = 0$.

For special case $u = 1$, simply

$$re^{i\theta} = \left(\frac{\cos\theta_k}{1 + \sin\theta_k}\right) i, \ r = \left|\frac{\cos\theta_k}{1 + \sin\theta_k}\right|, \ \theta = \frac{\pi}{2}(2 - \mathrm{sgn}(\cos\theta_k)) \tag{7.31}$$

where $\theta = \pi/2$ if $\cos\theta_k$ is positive and $\theta = 3\pi/2$ if $\cos\theta_k$ is negative.

We want to examine the relation between r, θ and u, θ_k as

$$\begin{aligned}
|\mathrm{Even}\rangle &= \frac{1}{\sqrt{2(1 + e^{-2|\beta|^2})}}(|\beta\rangle + |-\beta\rangle) \\
|\mathrm{Odd}\rangle &= \frac{1}{i\sqrt{2(1 - e^{-2|\beta|^2})}}(|\beta\rangle - |-\beta\rangle)
\end{aligned} \tag{7.32}$$

with overlap factor $e^{-2|\beta|^2}$. For general NPC state $c\left(|\mathrm{Even}\rangle + ue^{i\theta_k}|\mathrm{Odd}\rangle\right)$ with normalization condition

$$1 = |c|^2\left(1 + u^2\right), \quad |c| = \frac{1}{\sqrt{1 + u^2}} \tag{7.33}$$

$$
\begin{aligned}
&c\left(|\mathrm{Even}\rangle + ue^{i\theta_k}\,|\mathrm{Odd}\rangle\right) \\
&= c\left(\frac{1}{\sqrt{2(1+e^{-2|\beta|^2})}}\left(|\beta\rangle + |-\beta\rangle\right) + \frac{ue^{i\theta_k}}{i\sqrt{2(1-e^{-2|\beta|^2})}}\left(|\beta\rangle - |-\beta\rangle\right)\right) \\
&= \frac{c\left(1-u\lambda_\beta e^{i(\theta_k+\pi/2)}\right)}{\sqrt{2(1+e^{-2|\beta|^2})}}\left(|\beta\rangle + \frac{1+u\lambda_\beta e^{i(\theta_k+\pi/2)}}{1-u\lambda_\beta e^{i(\theta_k+\pi/2)}}\,|-\beta\rangle\right)
\end{aligned}
\tag{7.34}
$$

and

$$
r = \left|\frac{1+u\lambda_\beta e^{i(\theta_k+\pi/2)}}{1-u\lambda_\beta e^{i(\theta_k+\pi/2)}}\right|, \quad \theta = \mathrm{Arg}\left(\frac{1+u\lambda_\beta e^{i(\theta_k+\pi/2)}}{1-u\lambda_\beta e^{i(\theta_k+\pi/2)}}\right). \tag{7.35}
$$

where

$$
\lambda_\beta = \sqrt{\frac{1+e^{-2|\beta|^2}}{1-e^{-2|\beta|^2}}} \tag{7.36}
$$

and we see that by $u \to u\lambda_\beta$ we have the same expression as $e^{-2|\beta|^2} \simeq 0$ limit. Therefore we get

$$
\begin{aligned}
r &= \left|\frac{(1-u^2\lambda_\beta^2) + 2iu\lambda_\beta\cos\theta_k}{(1+u^2\lambda_\beta^2) + 2u\lambda_\beta\sin\theta_k}\right| \\
&= \sqrt{\frac{1+u^4\lambda_\beta^4 + 2u^2\lambda_\beta^2(\cos^2\theta_k - \sin^2\theta_k)}{(1+u^2\lambda_\beta^2)^2 + 4u^2\lambda_\beta^2\sin^2\theta_k + 4(1+u^2\lambda_\beta^2)u\lambda_\beta\sin\theta_k}}
\end{aligned}
\tag{7.37}
$$

and

$$
\theta = \mathrm{Arg}\left((1-u^2\lambda_\beta^2) + 2iu\lambda_\beta\cos\theta_k\right) = \tan^{-1}\left(\frac{2u\lambda_\beta\cos\theta_k}{1-u^2\lambda_\beta^2}\right) \tag{7.38}
$$

again $u = 0$ and $u \to \infty$ leads to $\theta = 0$.

Here we find a quantity

$$
|c|^2 u\sin\theta_k = \frac{u}{1+u^2}\sin\theta_k \tag{7.39}
$$

first in terms of r, θ within $e^{-2|\beta|^2} \simeq 0$ limit ($\lambda_\beta = 1$). $\langle \hat{a}_0^\dagger \hat{a}_1 \rangle, \langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_1 \rangle$, $\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \rangle$ are proportional to $|c|^2 u\sin\theta_k$ as will be shown later. Now it is easy to calculate that

$$\frac{1-r^2}{2(1+r^2)} = \frac{u}{1+u^2}\sin\theta_k \tag{7.40}$$

therefore $e^{-2|\beta|^2} \simeq 0$ limit yields that $\langle \hat{a}_0^\dagger \hat{a}_1 \rangle, \langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_1 \rangle, \langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \rangle$ depends on r but not on θ.

If we consider $e^{-2|\beta|^2}$ explicitly, we have

$$\frac{1-r^2}{2(1+r^2)} = \frac{u\lambda_\beta}{1+u^2\lambda_\beta^2}\sin\theta_k = \left(\frac{u}{1+u^2}\sin\theta_k\right)\frac{1+u^2}{\lambda_\beta(\lambda_\beta^{-2}+u^2)} \tag{7.41}$$

and since $u = u(r,\theta)$, as long as overlap between $|\beta\rangle$ and $|-\beta\rangle$ is significant therefore λ_β is different from 1, $|c|^2 u \sin\theta_k$ has θ dependence now.

7.2.1 Density-Density Correlation in $|\beta\rangle$ Basis

Until now, analogy was identified between fragmented state (generalized up to NPC state) and superposition of coherent state, or cat state. Now It is possible to analyze peculiar density-density correlation in single-trap fragmented state. Fig.7.2 plots $\rho_2(z,z')$ for $|-\beta\rangle$ (left, call it 'Dead cat'), $\frac{1}{\sqrt{2}}(|-\beta\rangle + e^{i\theta}|\beta\rangle)$ (center) and $|\beta\rangle$ [36] (right, call it 'Alive cat'). It is clear that central figure is simple average of left and right figures. Considering that $\langle -\beta | \beta\rangle \simeq 0$, it can be concluded that large fluctuation and following strong suppression along $z = -z'$ originates from macroscopic superposition of two states each biased to positive z direction and negative z direction. Density $\rho(z)$ captures only average of two 'cats', but macroscopic superposition actually does not allow correlation between positive and negative z sides. This 'fake' correlation in $\rho(z,t)\rho(z',t)$ was source of large fluctuation $\rho_2(z,z',t) - \rho(z,t)\rho(z',t)$.

Above illustration about fragmented state provides future problems.

1. What will single shot pattern be for fragmented state? Just two $|\beta\rangle$ and $|-\beta\rangle$ with equal probability?

2. Deviation of two 'cats' after TOF originates from that $|\beta\rangle$ and $|-\beta\rangle$ each correspond to states with biased momentum to left side and right side at $t = 0$.

3. Further investigation on analogy, e.g. negative probability of Wigner function will also happen for fragmented state with quadrature of $\hat{b}$?

These problems is going to be stated in conclusion chapter again.

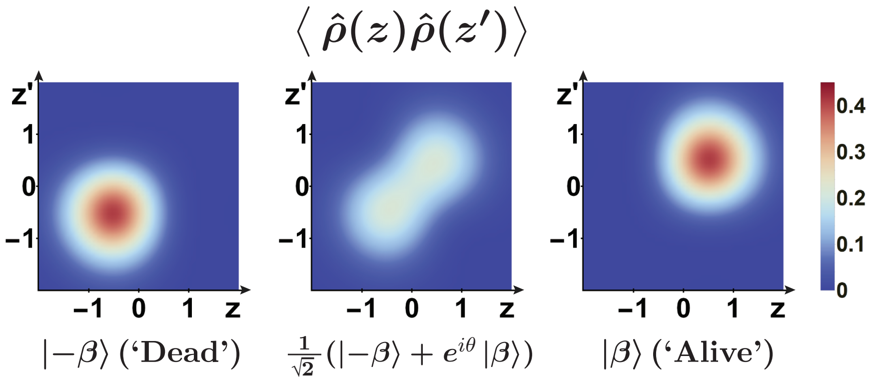

Figure 7.2: $\rho_2(z, z', t)$ after TOF for $|-\beta\rangle$ (left), $\frac{1}{\sqrt{2}}(|-\beta\rangle + e^{i\theta}|\beta\rangle)$ (center) and $|\beta\rangle$ (right)

7.3 A Relation Between Quadrature Fluctuation and Density-Density Correlation

Now we consider following superposition of $|\beta\rangle$ and $|-\beta\rangle$

$$\frac{1}{\sqrt{2(1+\cos\theta e^{-2|\beta|^2})}}\Big(|\beta\rangle + e^{i\theta}|-\beta\rangle\Big) \tag{7.42}$$

which corresponds to $r = 1$ case of (7.22) with $\beta = |\beta|e^{i\phi_\beta}$ and a value of ϕ_β varies $0 \sim 2\pi$. If N is large enough with small enough $|\beta|^2/N$, e.g. $N > 100$ and $|\beta|^2 < 60$. Then we have

$$\hat{b}\,|\beta\rangle \simeq \beta\,|\beta\rangle\,, \quad \text{and} \quad \left[\hat{b}, \hat{b}^\dagger\right] \simeq 1 \tag{7.43}$$

Further when $|\beta|^2 = 3$ (irrelevant of N)

$$\frac{\langle\beta|-\beta\rangle}{\langle\beta|\,\beta\rangle} < 0.003 \tag{7.44}$$

and decrease exponentially following $e^{-2|\beta|^2}$, thus it is safe to put $\langle\beta|-\beta\rangle \simeq 0$ if $|\beta|^2 > 3$ but we do not neglect $\langle\beta|-\beta\rangle$ here. We get an expectation value of $:\hat{O}(\hat{b}, \hat{b}^\dagger):$ for the state in (7.42) as

$$\left\langle :\hat{O}(\hat{b},\hat{b}^\dagger): \right\rangle = \frac{1}{2(1+\cos\theta e^{-2|\beta|^2})} \times \left(:\hat{O}(\beta,\beta^*): + :\hat{O}(-\beta,-\beta^*): + e^{i\theta}:\hat{O}(-\beta,\beta^*): + e^{-i\theta}:\hat{O}(\beta,-\beta^*): \right). \tag{7.45}$$

We may define quadratures $\frac{1}{\sqrt{2}}(\hat{b}+\hat{b}^\dagger)$ and $\frac{1}{\sqrt{2}i}(\hat{b}-\hat{b}^\dagger)$ following those of photon cat state. And expectation value over (7.42) of quadratures and squares of them are

$$\begin{aligned}
\left\langle\frac{1}{\sqrt{2}}(\hat{b}+\hat{b}^\dagger)\right\rangle &= \frac{\sin\theta e^{-2|\beta|^2}}{1+\cos\theta e^{-2|\beta|^2}}\sqrt{2}|\beta|\sin\phi_\beta \\
\left\langle\frac{1}{\sqrt{2}i}(\hat{b}-\hat{b}^\dagger)\right\rangle &= -\frac{\sin\theta e^{-2|\beta|^2}}{1+\cos\theta e^{-2|\beta|^2}}\sqrt{2}|\beta|\cos\phi_\beta \\
\left\langle\frac{1}{2}(\hat{b}+\hat{b}^\dagger)^2\right\rangle &= 2|\beta|^2\cos^2\phi_\beta + \frac{1}{2} - \frac{2|\beta|^2\cos\theta e^{-2|\beta|^2}}{1+\cos\theta e^{-2|\beta|^2}} \\
-\left\langle\frac{1}{2}(\hat{b}-\hat{b}^\dagger)^2\right\rangle &= 2|\beta|^2\sin^2\phi_\beta + \frac{1}{2} - \frac{2|\beta|^2\cos\theta e^{-2|\beta|^2}}{1+\cos\theta e^{-2|\beta|^2}}
\end{aligned} \tag{7.46}$$

and fluctuations are given as

$$\begin{aligned}
&\frac{1}{2}\left\langle(\hat{b}+\hat{b}^\dagger)^2\right\rangle - \left(\frac{1}{2}\left\langle(\hat{b}+\hat{b}^\dagger)\right\rangle\left\langle(\hat{b}+\hat{b}^\dagger)\right\rangle\right) \\
&= 2|\beta|^2\cos^2\phi_\beta + \frac{1}{2} - \frac{2|\beta|^2\cos\theta e^{-2|\beta|^2}}{1+\cos\theta e^{-2|\beta|^2}} - \frac{2|\beta|^2\sin^2\theta\sin^2\phi_\beta e^{-4|\beta|^2}}{(1+\cos\theta e^{-2|\beta|^2})^2} \\
&-\frac{1}{2}\left\langle(\hat{b}-\hat{b}^\dagger)^2\right\rangle - \left(-\frac{1}{2}\left\langle(\hat{b}-\hat{b}^\dagger)\right\rangle\left\langle(\hat{b}-\hat{b}^\dagger)\right\rangle\right) \\
&= -2|\beta|^2\sin^2\phi_\beta + \frac{1}{2} - \frac{2|\beta|^2\cos\theta e^{-2|\beta|^2}}{1+\cos\theta e^{-2|\beta|^2}} - \frac{2|\beta|^2\sin^2\theta\cos^2\phi_\beta e^{-4|\beta|^2}}{(1+\cos\theta e^{-2|\beta|^2})^2}
\end{aligned} \tag{7.47}$$

therefore showing huge fluctuation of order of $\left\langle\hat{b}^\dagger\hat{b}\right\rangle = |\beta|^2$ in second moments of $\frac{1}{\sqrt{2}}(\hat{b}+\hat{b}^\dagger)$ and $\frac{1}{\sqrt{2}i}(\hat{b}-\hat{b}^\dagger)$ depending on the value of ϕ_β. $\phi_\beta = 0$ corresponds to large fluctuation in $\frac{1}{\sqrt{2}}(\hat{b}+\hat{b}^\dagger)$ which is analogous to position $\hat{x} = \frac{1}{\sqrt{2}}(\hat{a}+\hat{a}^\dagger)$ in photon cat state and $\phi_\beta = \pi/2$ corresponds to large fluctuation in $\frac{1}{\sqrt{2}i}(\hat{b}-\hat{b}^\dagger)$ which is analogous to momentum $\hat{p} = \frac{1}{\sqrt{2}i}(\hat{a}-\hat{a}^\dagger)$ in photon cat state. In terms of $\hat{a}_0, \hat{a}_0^\dagger, \hat{a}_1, \hat{a}_1^\dagger$, $\frac{1}{\sqrt{2}}(\hat{b}+\hat{b}^\dagger)$ and $\frac{1}{\sqrt{2}i}(\hat{b}-\hat{b}^\dagger)$ are written as

$$\begin{aligned}\frac{1}{\sqrt{2}}(\hat{b}+\hat{b}^\dagger) &= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{\hat{N}_0}}\hat{a}_0^\dagger\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_0\frac{1}{\sqrt{\hat{N}_0}}\right) \\ \frac{1}{\sqrt{2}i}(\hat{b}-\hat{b}^\dagger) &= \frac{1}{\sqrt{2}i}\left(\frac{1}{\sqrt{\hat{N}_0}}\hat{a}_0^\dagger\hat{a}_1 - \hat{a}_1^\dagger\hat{a}_0\frac{1}{\sqrt{\hat{N}_0}}\right)\end{aligned} \tag{7.48}$$

We ignored $\epsilon \to 0$ in original definition of $\hat{b}, \hat{b}^\dagger$ since $\epsilon \to 0$ was introduced to eliminate divergence happening for Fock state $|0, N\rangle$, and we are not interested in $|0, N\rangle$ here. Now expectation values of square of $\frac{1}{\sqrt{2}}(\hat{b}+\hat{b}^\dagger)$ is

$$\begin{aligned}(\hat{b}+\hat{b}^\dagger)^2 = &\left(\frac{1}{\sqrt{\hat{N}_0}}\hat{a}_0^\dagger\hat{a}_1\frac{1}{\sqrt{\hat{N}_0}}\hat{a}_0^\dagger\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_0\frac{1}{\sqrt{\hat{N}_0}}\hat{a}_1^\dagger\hat{a}_0\frac{1}{\sqrt{\hat{N}_0}}\right) \\ &+\left(\frac{1}{\sqrt{\hat{N}_0}}\hat{a}_0^\dagger\hat{a}_1\hat{a}_1^\dagger\hat{a}_0\frac{1}{\sqrt{\hat{N}_0}} + \hat{a}_1^\dagger\hat{a}_0\frac{1}{\hat{N}_0}\hat{a}_0^\dagger\hat{a}_1\right)\end{aligned} \tag{7.49}$$

and $\frac{1}{\sqrt{2}i}(\hat{b}-\hat{b}^\dagger)$ are

$$\begin{aligned}-(\hat{b}-\hat{b}^\dagger)^2 = -&\left(\frac{1}{\sqrt{\hat{N}_0}}\hat{a}_0^\dagger\hat{a}_1\frac{1}{\sqrt{\hat{N}_0}}\hat{a}_0^\dagger\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_0\frac{1}{\sqrt{\hat{N}_0}}\hat{a}_1^\dagger\hat{a}_0\frac{1}{\sqrt{\hat{N}_0}}\right) \\ &+\left(\frac{1}{\sqrt{\hat{N}_0}}\hat{a}_0^\dagger\hat{a}_1\hat{a}_1^\dagger\hat{a}_0\frac{1}{\sqrt{\hat{N}_0}} + \hat{a}_1^\dagger\hat{a}_0\frac{1}{\hat{N}_0}\hat{a}_0^\dagger\hat{a}_1\right).\end{aligned} \tag{7.50}$$

neglecting small error which is relatively $\mathcal{O}(1/ < \hat{N}_0 >)$ coming from commutation, we have

$$\begin{aligned}\frac{1}{2}\langle(\hat{b}+\hat{b}^\dagger)^2\rangle &= \Re\left[\langle\frac{1}{\hat{N}_0}\left(\hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0 + \hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_1\hat{a}_1\right)\rangle\right] \\ -\frac{1}{2}\langle(\hat{b}-\hat{b}^\dagger)^2\rangle &= \Re\left[\langle\frac{1}{\hat{N}_0}\left(\hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0 - \hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_1\hat{a}_1\right)\rangle\right].\end{aligned} \tag{7.51}$$

Now we are going to deal with density-density correlation $\rho_2(z, z')$ between z and z' of 1-D NPC state which describes fragmented state in a single trap in [22]. This state can be expressed as (7.42) in terms of $|\beta\rangle$ basis as long as σ^2 of $|C_l|$ distribution of NPC state is smaller than $|\beta|^2$. Here we fixed as $|\beta|^2 = l_0$

where l_0 is center of $|C_l|$ distribution of NPC state.

$$\begin{aligned}\rho_2(z,z') =&\langle\hat{\psi}^\dagger(z)\hat{\psi}^\dagger(z')\hat{\psi}(z')\hat{\psi}(z)\rangle = |\psi_0(z)|^2|\psi_0(z')|^2\langle\hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_0\hat{a}_0\rangle \\ &+ 0\to 1 + \left(|\psi_0(z)|^2|\psi_1(z')|^2 + 0\leftrightarrow 1\right)\langle\hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0\rangle \\ &+ 2\Re\Big[\psi_0^*(z)\psi_1^*(z')\psi_0(z')\psi_1(z)\langle\hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0\rangle \\ &+\psi_0^*(z)\psi_0^*(z')\psi_1(z')\psi_1(z)\langle\hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_1\hat{a}_1\rangle\Big].\end{aligned} \tag{7.52}$$

where $\psi_i(z)$ denote for orbitals of each i-th mode where in two-mode approximation where field operator $\hat{\psi}(z)$ is truncated as $\hat{\psi}(z) \approx \psi_0(z)\hat{a}_0 + \psi_1(z)\hat{a}_1$. Here (in [22]) $\psi_0(z)$ is an even function of z and $\psi_1(z)$ is an odd function of z as follows

$$\psi_0(-z) = \psi_0(z), \quad \psi_1(-z) = -\psi_1(z) \tag{7.53}$$

with $\psi_0(z), \psi_1(z) \in \mathbb{R}$. Under this even-odd parity of $\psi_0(z)$ and $\psi_1(z)$, $\rho_2(z,-z)$ becomes

$$\begin{aligned}&\rho_2(z,-z) = |\psi_0(z)|^4\langle\hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_0\hat{a}_0\rangle + |\psi_1(z)|^4\langle\hat{a}_1^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_1\rangle \\ &+2|\psi_0(z)|^2|\psi_1(z)|^2\langle\hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0\rangle - 2|\psi_0(z)|^2|\psi_1(z)|^2\,\Re\left[\langle\hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0 + \hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_1\hat{a}_1\rangle\right].\end{aligned} \tag{7.54}$$

And this can be simplified further considering that $\langle\hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_0\hat{a}_0\rangle, \langle\hat{a}_1^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_1\rangle$, $\langle\hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0\rangle$ are obtained from $|C_l|$ distribution irrelevant of ϕ_l which is phase of C_l. For example, in the case of $\langle\hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_0\hat{a}_0\rangle$

$$\langle\hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_0\hat{a}_0\rangle = \sum_{l=0}^{N-2}|C_l|^2(N-l)(N-l-1) \tag{7.55}$$

Thus for Gaussian $|C_l|$ distribution, ignoring fluctuation we have up to $\mathcal{O}(1/l_0)$ and $\mathcal{O}(1/(N-l_0))$

$$\langle\hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_0\hat{a}_0\rangle = (N-l_0)^2,\ \langle\hat{a}_1^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_1\rangle = l_0^2,\ \langle\hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0\rangle = (N-l_0)l_0 \tag{7.56}$$

with

$$\langle\hat{a}_0^\dagger\hat{a}_0\rangle = N-l_0,\ \langle\hat{a}_1^\dagger\hat{a}_1\rangle = l_0 \tag{7.57}$$

where l_0 is center of Gaussian $|C_l|$ distribution of the NPC state (under continuum limit) we are dealing with. Then (7.54) is greatly simplified as

$$\begin{aligned}\rho_2(z,-z) &\simeq \left((N-l_0)|\psi_0(z)|^2 + l_0|\psi_1(z)|^2\right)^2 \\ &\qquad - 2|\psi_0(z)|^2|\psi_1(z)|^2\ \Re\left[\left\langle \hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0 + \hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_1\hat{a}_1\right\rangle\right] \\ &= (\rho_1(z))^2 - 2|\psi_0(z)|^2|\psi_1(z)|^2\ \Re\left[\left\langle \hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0 + \hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_1\hat{a}_1\right\rangle\right]\end{aligned} \tag{7.58}$$

where $\rho_1(z) = \left\langle\hat{\psi}^\dagger(z)\hat{\psi}(z)\right\rangle$ is a density profile at z. Therefore, density-density correlation can be expressed as a summation of square of density at z (and $\rho_1(-z) = \rho_1(z)$) and remaining part related to $\left\langle \hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0 + \hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_1\hat{a}_1\right\rangle$. This $\left\langle \hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0 + \hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_1\hat{a}_1\right\rangle$ part clearly contributes to fluctuation happens for density-density correlation between z and $-z$ and further is directly related to $\left\langle\frac{1}{2}(\hat{b}+\hat{b}^\dagger)^2\right\rangle$ in (7.51). Actually, $1/\hat{N}_0$ in (7.51) up to small error is evaluated as $1/(N-l_0)$ such that

$$\begin{aligned}\left\langle\frac{1}{2}(\hat{b}+\hat{b}^\dagger)^2\right\rangle &\simeq \frac{1}{N-l_0}\Re\left[\left\langle \hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0 + \hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_1\hat{a}_1\right\rangle\right] \\ \left\langle-\frac{1}{2}(\hat{b}-\hat{b}^\dagger)^2\right\rangle &\simeq \frac{1}{N-l_0}\Re\left[\left\langle \hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0 - \hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_1\hat{a}_1\right\rangle\right].\end{aligned} \tag{7.59}$$

Since NPC state corresponds to superposition of $|\beta\rangle$ and $|-\beta\rangle$ with $\phi_\beta = \pi/2$, fluctuation in density-density correlation can be expressed as $\frac{1}{2}(\hat{b}+\hat{b}^\dagger)^2$ which is a second moment of quadrature (which is also fluctuation of quadrature where $\left\langle\frac{1}{\sqrt{2}}(\hat{b}+\hat{b}^\dagger)\right\rangle = 0$) at $t=0$. And with $\phi_\beta = \pi/2$ we expect 0 fluctuation which exactly matches with result in [22].

And after a fragmented condensate in a single trap is released from the trap, therefore $t=0$ while condensate is in the trap, with Time-Of-Flight (TOF) after certain time $t \gg 1$ (which is a timescale condensate expands much larger than size of condensate at $t=0$) we have $\psi_i(z,t)$ as

$$\psi_i(z,t) = \sqrt{\frac{1}{2\pi i w_t^2}}\exp\left[\frac{iz^2}{2w_t^2}\right]\tilde{\psi}_i(z,t); \quad w_t = \sqrt{t}. \tag{7.60}$$

with time-invariant $\hat{a}_0, \hat{a}_1$ [17, 43] which means

$$\left\langle\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_k\hat{a}_l\right\rangle(t) = \left\langle\hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_k\hat{a}_l\right\rangle(t=0), \quad i,j,k,l = 0,1. \tag{7.61}$$

This time still even (odd) function remains as even (odd) function, but $\psi_1(z)$

acquires relative phase difference $\pi/2$ compared to $\psi_0(z)$. Therefore density-density correlation $\rho_2(z,-z,t)$ between z and $-z$ at time t is

$$\begin{aligned}\rho_2(z,-z,t) &\simeq \left((N-l_0)|\psi_0(z,t)|^2 + l_0|\psi_1(z,t)|^2\right)^2 \\ &\quad - 2|\psi_0(z,t)|^2|\psi_1(z,t)|^2\ \Re\left[\langle \hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0 - \hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_1\hat{a}_1\rangle\right] \\ &= (\rho_1(z,t))^2 - 2|\psi_0(z,t)|^2|\psi_1(z,t)|^2\ \Re\left[\langle \hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0 - \hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_1\hat{a}_1\rangle\right]\end{aligned} \tag{7.62}$$

and after TOF, now fluctuation in density-density correlation is directly related to $-\frac{1}{2}(\hat{b}-\hat{b}^\dagger)^2$ this time. And with $\phi_\beta = \pi/2$ we expect maximal fluctuation which also exactly matches with result in [22], therefore having suppression of density-density correlation between z and $-z$ after TOF.

To summarize, fluctuation of quadrature in $|\beta\rangle$ basis which is approximate coherent state basis has a direct relation to fluctuation in density-density correlation of NPC state in (7.42) between z and $-z$ with even-odd parity of $\psi_0(z), \psi_1(z)$. Or, fluctuation in density-density correlation of NPC state in (7.42) between z and $-z$ after TOF is directly proportional to fluctuation of quadrature in $|\beta\rangle$ basis where $\phi_\beta = \pi/2$.

Now let us consider following general superposition of $|\beta\rangle$ and $|-\beta\rangle$,

$$\frac{1}{\sqrt{1+r^2+2r\cos\theta e^{-2|\beta|^2}}}\left(|\beta\rangle + re^{i\theta}\,|-\beta\rangle\right) \tag{7.63}$$

which corresponds to arbitrary r case of (7.22) with $\beta = |\beta|e^{i\phi_\beta}$ and a value of ϕ_β varies $0 \sim 2\pi$. If N is large enough with small enough $|\beta|^2/N$, e.g. $N > 100$ and $|\beta|^2 < 60$. Then we have

$$\hat{b}\,|\beta\rangle \simeq \beta\,|\beta\rangle\,, \quad \text{and } \left[\hat{b}, \hat{b}^\dagger\right] \simeq 1 \tag{7.64}$$

Further when $|\beta|^2 = 3$ (irrelevant of N)

$$\frac{\langle\beta|\ -\beta\rangle}{\langle\beta|\ \beta\rangle} < 0.003 \tag{7.65}$$

and decrease exponentially following $e^{-2|\beta|^2}$, thus it is safe to put $\langle\beta|\ -\beta\rangle \simeq 0$

for $|\beta|^2 > 3$. It means that we can put (7.63) as

$$\frac{1}{\sqrt{1+r^2}}\Big(|\beta\rangle + re^{i\theta}\,|-\beta\rangle\Big) \tag{7.66}$$

then in terms of $|\text{Even}\rangle$ and $|\text{Odd}\rangle$ basis

$$\frac{1}{\sqrt{1+u^2}}\Big(|\text{Even}\rangle + ue^{i\theta_k}\,|\text{Odd}\rangle\Big) \tag{7.67}$$

Here we calculate quadratures exactly using (7.63) instead of (7.66) first. We get an expectation value of $:\hat{O}(\hat{b},\hat{b}^\dagger):$ for the state in (7.63) as

$$\begin{aligned}\langle :\hat{O}(\hat{b},\hat{b}^\dagger):\rangle =& \frac{1}{1+r^2+2r\cos\theta e^{-2|\beta|^2}}\Big(:\hat{O}(\beta,\beta^*): + r^2:\hat{O}(-\beta,-\beta^*): \\ &+ re^{i\theta}e^{-2|\beta|^2}:\hat{O}(-\beta,\beta^*): + re^{-i\theta}e^{-2|\beta|^2}:\hat{O}(\beta,-\beta^*):\Big).\end{aligned} \tag{7.68}$$

We get quadratures and square of them for (7.66) (ignoring commutator between $\hat{b}$ and $\hat{b}^\dagger$) as

$$\begin{aligned}\langle\frac{1}{\sqrt{2}}(\hat{b}+\hat{b}^\dagger)\rangle =& \frac{1}{1+r^2+2r\cos\theta e^{-2|\beta|^2}}\sqrt{2}|\beta| \\ &\times\Big((1-r^2)\cos\phi_\beta + 2r\sin\theta\sin\phi_\beta\, e^{-2|\beta|^2}\Big) \\ \langle\frac{1}{\sqrt{2}i}(\hat{b}-\hat{b}^\dagger)\rangle =& \frac{1}{1+r^2+2r\cos\theta e^{-2|\beta|^2}}\sqrt{2}|\beta| \\ &\times\Big((1-r^2)\sin\phi_\beta - 2r\sin\theta\cos\phi_\beta\, e^{-2|\beta|^2}\Big) \\ \langle\frac{1}{2}(\hat{b}+\hat{b}^\dagger)^2\rangle =& 2|\beta|^2\cos^2\phi_\beta + \frac{1}{2} - \frac{4r|\beta|^2\cos\theta\, e^{-2|\beta|^2}}{1+r^2+2r\cos\theta\, e^{-2|\beta|^2}} \\ -\langle\frac{1}{2}(\hat{b}-\hat{b}^\dagger)^2\rangle =& 2|\beta|^2\sin^2\phi_\beta + \frac{1}{2} - \frac{4r|\beta|^2\cos\theta\, e^{-2|\beta|^2}}{1+r^2+2r\cos\theta\, e^{-2|\beta|^2}}\end{aligned} \tag{7.69}$$

with different quadratures from $r=1$ case but invariant square of quadratures.

Fluctuations are given as

$$
\begin{aligned}
&\frac{1}{2}\langle(\hat{b}+\hat{b}^\dagger)^2\rangle - \left(\frac{1}{2}\langle(\hat{b}+\hat{b}^\dagger)\rangle\langle(\hat{b}+\hat{b}^\dagger)\rangle\right)\\
=&2|\beta|^2\cos^2\phi_\beta + \frac{1}{2} - \frac{4r|\beta|^2\cos\theta\, e^{-2|\beta|^2}}{1+r^2+2r\cos\theta\, e^{-2|\beta|^2}}\\
&-2|\beta|^2\left(\frac{(1-r^2)\cos\phi_\beta + 2r\sin\theta\sin\phi_\beta\, e^{-2|\beta|^2}}{1+r^2+2r\cos\theta e^{-2|\beta|^2}}\right)^2\\
&-\frac{1}{2}\langle(\hat{b}-\hat{b}^\dagger)^2\rangle - \left(-\frac{1}{2}\langle(\hat{b}-\hat{b}^\dagger)\rangle\langle(\hat{b}-\hat{b}^\dagger)\rangle\right)\\
=&2|\beta|^2\sin^2\phi_\beta + \frac{1}{2} - \frac{4r|\beta|^2\cos\theta\, e^{-2|\beta|^2}}{1+r^2+2r\cos\theta\, e^{-2|\beta|^2}}\\
&-2|\beta|^2\left(\frac{(1-r^2)\sin\phi_\beta - 2r\sin\theta\cos\phi_\beta\, e^{-2|\beta|^2}}{1+r^2+2r\cos\theta e^{-2|\beta|^2}}\right)^2
\end{aligned}
\tag{7.70}
$$

which gives for $r = 0$ (which corresponds to $|\beta\rangle$) and for $r \to \infty$ (which corresponds to $|-\beta\rangle$) almost 0 fluctuation except $\frac{1}{2}$ comes from commutator of $\hat{b}$ and $\hat{b}^\dagger$. Here quantum shot noise term 1/2 is written to show that as long as $|\beta|^2 \gg 1$ shot noise is negligible one.

For density-density correlation in [22], we consider also more general case, which NPC state with extra degree of freedom θ_k we have been used which is introduced in [21] (where θ was used, but here we instead use θ_k to prevent confusion with θ which is already defined). Then we have full degree of freedom for positive u and θ_k for $|\mathrm{Even}\rangle$ and $|\mathrm{Odd}\rangle$ basis. And also full degree of freedom for positive r and θ in $|\beta\rangle$ basis is achieved, but we have fixed $\phi_\beta = \pi/2$ now with negative pair coherence. And also, we ignore $\langle -\beta|\ \beta\rangle$ therefore an effect of θ is ignored here in calculation of density-density correlation function $\rho_2(z,z')$.

This time, we have additional $\left\langle \hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_0\hat{a}_1\right\rangle$ and $\left\langle \hat{a}_1^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0\right\rangle$ (and their Hermitian conjugate, also) for $\rho_2(z,z')$ which vanishes for $r=1$ case.

In two-mode approximation, density at z $\rho_1(z) = \left\langle\hat{\psi}^\dagger(z)\hat{\psi}(z)\right\rangle$ is written as

$$
\rho_1(z) = |\psi_0(z)|^2\langle\hat{a}_0^\dagger\hat{a}_0\rangle + |\psi_1(z)|^2\langle\hat{a}_1^\dagger\hat{a}_1\rangle + 2\Re\left[\psi_0^*(z)\psi_1(z)\langle\hat{a}_0^\dagger\hat{a}_1\rangle\right] \tag{7.71}
$$

and since $\psi_0(z), \psi_1(z) \in \mathbb{R}$

$$\rho_1(z) = |\psi_0(z)|^2 \langle \hat{a}_0^\dagger \hat{a}_0 \rangle + |\psi_1(z)|^2 \langle \hat{a}_1^\dagger \hat{a}_1 \rangle + 2\psi_0(z)\psi_1(z) \Re \left[\langle \hat{a}_0^\dagger \hat{a}_1 \rangle \right]. \tag{7.72}$$

Density-density correlation $\rho_2(z, z')$ is written as

$$\begin{aligned}
\rho_2(z, z') = &|\psi_0(z)|^2 |\psi_0(z')|^2 \langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 \rangle + |\psi_1(z)|^2 |\psi_1(z')|^2 \langle \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \rangle \\
&+ \left(|\psi_0(z)|^2 |\psi_1(z')|^2 + |\psi_0(z')|^2 |\psi_1(z)|^2 \right) \langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 \rangle \\
&+ \left(\psi_1^*(z)\psi_0^*(z')\psi_1(z')\psi_0(z) + \psi_0^*(z)\psi_1^*(z')\psi_0(z')\psi_1(z) \right) \langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 \rangle \\
&+ \left(\psi_0^*(z)\psi_0^*(z')\psi_1(z')\psi_1(z) \langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \rangle + \text{h.c.} \right) \\
&+ \left(\psi_0^*(z)\psi_0^*(z')(\psi_1(z')\psi_0(z) + \psi_0(z')\psi_1(z)) \langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_1 \rangle + \text{h.c.} \right) \\
&+ \left(\psi_1(z)\psi_1(z')(\psi_0^*(z')\psi_1^*(z) + \psi_1^*(z')\psi_0^*(z)) \langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \rangle + \text{h.c.} \right)
\end{aligned} \tag{7.73}$$

For $z' = -z$, we have $\rho_2(z, -z)$ as

$$\begin{aligned}
&\rho_2(z, -z) = \\
&|\psi_0(z)|^4 \langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 \rangle + |\psi_1(z)|^4 \langle \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \rangle + 2|\psi_0(z)|^2 |\psi_1(z)|^2 \langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 \rangle \\
&- 2|\psi_0(z)|^2 |\psi_1(z)|^2 \ \Re \left[\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 + \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \rangle \right]
\end{aligned} \tag{7.74}$$

which is the same as $r = 1$ case. This is due to even-odd parity of $\psi_0(z)$ and $\psi_1(z)$, which does not allow terms related to $\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_1 \rangle$ and $\langle \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 \rangle$. This time, we also have a look at $\rho_2(z, z)$ which will play a crucial role.

$$\begin{aligned}
&\rho_2(z, z) = |\psi_0(z)|^4 \langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 \rangle + |\psi_1(z)|^4 \langle \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \rangle \\
&+ 2|\psi_0(z)|^2 |\psi_1(z)|^2 \langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 \rangle \\
&+ 2|\psi_0(z)|^2 |\psi_1(z)|^2 \ \Re \left[\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 + \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \rangle \right] \\
&+ 4|\psi_0(z)|^2 \psi_0(z)\psi_1(z) \Re \left[\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_1 \rangle \right] + 4|\psi_1(z)|^2 \psi_0(z)\psi_1(z) \Re \left[\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \rangle \right]
\end{aligned} \tag{7.75}$$

For Gaussian $|C_l|$ distribution, $\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_1 \rangle$, $\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \rangle$ can be expressed in terms of $\langle \hat{a}_0^\dagger \hat{a}_1 \rangle$, $\langle \hat{a}_0^\dagger \hat{a}_0 \rangle = N - l_0$ and $\langle \hat{a}_1^\dagger \hat{a}_1 \rangle = l_0$ as

$$\begin{aligned} \langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_1 \rangle &\simeq \langle \hat{a}_0^\dagger \hat{a}_0 \rangle \langle \hat{a}_0^\dagger \hat{a}_1 \rangle = (N - l_0) \langle \hat{a}_0^\dagger \hat{a}_1 \rangle \\ \langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_1 \rangle &\simeq \langle \hat{a}_0^\dagger \hat{a}_1 \rangle \langle \hat{a}_1^\dagger \hat{a}_1 \rangle = l_0 \langle \hat{a}_0^\dagger \hat{a}_1 \rangle \end{aligned} \tag{7.76}$$

up to an error which comes from standard deviation of $|C_l|$ distribution related to 'width' of the distribution. After some labor,

$$\begin{aligned} &|\psi_0(z)|^4 \langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 \rangle + |\psi_1(z)|^4 \langle \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \rangle + 2|\psi_0(z)|^2 |\psi_1(z)|^2 \langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 \rangle \\ &\simeq \Big(|\psi_0(z)|^2 \langle \hat{a}_0^\dagger \hat{a}_0 \rangle + |\psi_1(z)|^2 \langle \hat{a}_1^\dagger \hat{a}_1 \rangle \Big)^2 \end{aligned} \tag{7.77}$$

and as done in (7.62) we have fluctuations each $\rho_2(z,-z) - \rho_1(z)\rho_1(-z)$ and $\rho_2(z,z) - \rho_1(z)\rho_1(z)$ as

$$\begin{aligned} &\rho_2(z,-z) - \rho_1(z)\rho_1(-z) \simeq \\ &-4|\psi_0(z)|^2 |\psi_1(z)|^2 \left(\frac{1}{2} \Re \left[\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 + \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \rangle \right] - \left(\Re \left[\langle \hat{a}_0^\dagger \hat{a}_1 \rangle \right] \right)^2 \right) \end{aligned} \tag{7.78}$$

and

$$\begin{aligned} &\rho_2(z,z) - \rho_1(z)\rho_1(z) \simeq \\ &4|\psi_0(z)|^2 |\psi_1(z)|^2 \left(\frac{1}{2} \Re \left[\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 + \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \rangle \right] - \left(\Re \left[\langle \hat{a}_0^\dagger \hat{a}_1 \rangle \right] \right)^2 \right) \end{aligned} \tag{7.79}$$

And from

$$\begin{aligned} &\langle \hat{a}_0^\dagger \hat{a}_1 \rangle = \langle \sqrt{\hat{N}_0}\, \hat{b} \rangle, \ \Re[\langle \hat{a}_0^\dagger \hat{a}_1 \rangle] \simeq \frac{1}{2} \langle \sqrt{\hat{N}_0}\, (\hat{b} + \hat{b}^\dagger) \rangle \\ &\Im[\langle \hat{a}_0^\dagger \hat{a}_1 \rangle] \simeq \frac{1}{2i} \langle \sqrt{\hat{N}_0}\, (\hat{b} - \hat{b}^\dagger) \rangle \end{aligned} \tag{7.80}$$

with (7.51) we see that

$$\begin{aligned} &\rho_2(z,-z) - \rho_1(z)\rho_1(-z) \simeq \\ &-|\psi_0(z)|^2 |\psi_1(z)|^2 \left(\langle \left(\sqrt{\hat{N}_0}\, (\hat{b} + \hat{b}^\dagger) \right)^2 \rangle - \left(\langle \sqrt{\hat{N}_0}\, (\hat{b} + \hat{b}^\dagger) \rangle \right)^2 \right) \end{aligned} \tag{7.81}$$

and

$$\rho_2(z,z) - \rho_1(z)\rho_1(z) \simeq$$
$$|\psi_0(z)|^2|\psi_1(z)|^2 \left(\left\langle \left(\sqrt{\hat{N}_0}\,(\hat{b}+\hat{b}^\dagger) \right)^2 \right\rangle - \left(\left\langle \sqrt{\hat{N}_0}\,(\hat{b}+\hat{b}^\dagger) \right\rangle \right)^2 \right) \tag{7.82}$$

thus fluctuation of quadrature is directly related to fluctuation in density-density correlation with respect to square of density. And this decrease with non vanishing $\Re[\langle \hat{a}_0^\dagger \hat{a}_1 \rangle]$ and $\Im[\langle \hat{a}_0^\dagger \hat{a}_1 \rangle]$ for $r \neq 1$ as shown in (7.69) and (7.70).

After TOF, still $\psi_0(z,t) = \psi_0(-z,t), \psi_1(z,t) = -\psi_1(-z,t)$ with (see appendix B)

$$i\psi_1(z,t) \equiv \bar{\psi}_1(z,t), \quad |\bar{\psi}_1(z,t)|^2 = |\psi_1(z,t)|^2 \tag{7.83}$$

and $\psi_0^*(z,t)\bar{\psi}_1(z,t) \in \mathbb{R}$ where $\psi_0(z,t)$ and $\psi_1(z,t)$ share a phase factor which depend on position and time. Then we have $\rho_1(z,t)$ as

$$\rho_1(z,t) = |\psi_0(z,t)|^2\langle \hat{a}_0^\dagger \hat{a}_0 \rangle + |\psi_1(z,t)|^2\langle \hat{a}_1^\dagger \hat{a}_1 \rangle - 2\psi_0(z,t)\bar{\psi}_1(z,t)\Im\left[\langle \hat{a}_0^\dagger \hat{a}_1 \rangle\right]. \tag{7.84}$$

$\rho_2(z,-z,t)$ is

$$\begin{aligned}\rho_2(z,-z,t) = {} & |\psi_0(z,t)|^4\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 \rangle + |\psi_1(z,t)|^4\langle \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \rangle \\ & + 2|\psi_0(z,t)|^2|\psi_1(z,t)|^2\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 \rangle \\ & - 2|\psi_0(z,t)|^2|\psi_1(z,t)|^2\ \Re\left[\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 - \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \rangle\right].\end{aligned} \tag{7.85}$$

which is of the same as $r = 1$ case. This is because even-odd parity cancels terms related to $\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_1 \rangle$ and $\langle \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 \rangle$ for $z' = -z$. And fluctuations each $\rho_2(z,-z,t) - \rho_1(z,t)\rho_1(-z,t)$ and $\rho_2(z,z,t) - \rho_1(z,t)\rho_1(z,t)$ are

$$\rho_2(z,-z,t) - \rho_1(z,t)\rho_1(-z,t) \simeq$$
$$-4|\psi_0(z,t)|^2|\psi_1(z,t)|^2 \left(\frac{1}{2}\Re\left[\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0 - \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \rangle\right] - \left(\Im\left[\langle \hat{a}_0^\dagger \hat{a}_1 \rangle\right] \right)^2 \right) \tag{7.86}$$

and

$$\rho_2(z,z,t)-\rho_1(z,t)\rho_1(z,t)\simeq$$
$$4|\psi_0(z,t)|^2|\psi_1(z,t)|^2\left(\frac{1}{2}\Re\left[\langle\hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0-\hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_1\hat{a}_1\rangle\right]-\left(\Im\left[\langle\hat{a}_0^\dagger\hat{a}_1\rangle\right]\right)^2\right) \tag{7.87}$$

therefore in terms of $\hat{b},\hat{b}^\dagger$ we have

$$\rho_2(z,-z,t)-\rho_1(z,t)\rho_1(-z,t)\simeq$$
$$-2|\psi_0(z)|^2|\psi_1(z)|^2\left(\langle\left(\sqrt{\hat{N}_0}\frac{1}{\sqrt{2}i}(\hat{b}-\hat{b}^\dagger)\right)^2\rangle-\left(\langle\sqrt{\hat{N}_0}\frac{1}{\sqrt{2}i}(\hat{b}-\hat{b}^\dagger)\rangle\right)^2\right) \tag{7.88}$$

and

$$\rho_2(z,z,t)-\rho_1(z,t)\rho_1(z,t)\simeq$$
$$2|\psi_0(z)|^2|\psi_1(z)|^2\left(\langle\left(\sqrt{\hat{N}_0}\frac{1}{\sqrt{2}i}(\hat{b}-\hat{b}^\dagger)\right)^2\rangle-\left(\langle\sqrt{\hat{N}_0}\frac{1}{\sqrt{2}i}(\hat{b}-\hat{b}^\dagger)\rangle\right)^2\right) \tag{7.89}$$

which is also related to fluctuation of quadrature directly.

In conclusion, we see how both density ρ_1 and density-density correlation ρ_2 depends on quadrature directly, and as r goes far away from 1 (which can be quantified from $(1-r^2)/(1+r^2)$) fluctuation in density-density correlation decreases which directly follows decreases of fluctuation of quadrature. And we also can say that non-vanishing $\langle\hat{a}_0^\dagger\hat{a}_1\rangle$, which increases as $r\to 0$ or $r\to\infty$ following $(1-r^2)/(1+r^2)$, suppresses fluctuations. This can be interpreted as consequence of superposition of two approximate coherent state $|\beta\rangle$ and $|-\beta\rangle$ approaches to one of $|\beta\rangle$ or $|-\beta\rangle$ which is coherent therefore suppresses fluctuation to minimum.

Further, fluctuation above only ‘reduces’ density-density correlation between z and $-z$ but at the same time it only ‘enhances’ density-density correlation at z. And we would like to note that, with NPC state used in [8, 21, 22] non 0 θ_k, which is introduced in [21], tends to decrease fluctuation letting r be far from $r=1$. And from (7.69) and (7.70) we also see that when we consider general superposition of $|\beta\rangle$ and $|-\beta\rangle$ where ϕ_β varies from 0 to 2π, depending on the value of ϕ_β it is determined that whether we can see the fluctuation before TOF or after TOF or at both.

Now we are going to calculate following fluctuation in density $\Delta\rho_2(z, z')$ exactly considering commutator and for general z, z'

$$\Delta\rho_2(z, z') \equiv \big\langle \hat{\rho}_1(z)\hat{\rho}_1(z')\big\rangle - \big\langle \hat{\rho}_1(z)\big\rangle\big\langle \hat{\rho}_1(z')\big\rangle. \tag{7.90}$$

Before calculating $\big\langle \hat{\rho}_1(z)\hat{\rho}_1(z')\big\rangle$, we would like have a look at truncation of field operator $\hat{\psi}(z)$ as (see appendix A for detail)

$$\hat{\psi}(z) \approx \psi_0(z)\hat{a}_0 + \psi_1(z)\hat{a}_1 \tag{7.91}$$

where exact expansion of $\hat{\psi}(z)$ is

$$\hat{\psi}(z) = \sum_{i=0}^{\infty} \psi_i(z)\hat{a}_i \tag{7.92}$$

with $\{\psi_i(z)\}$ which is complete basis set of 1 particle (and 1 dimension, z) Hilbert space. And we want to point out that applying two-mode approximation simply with (7.91) fails when we calculate for certain kind of operators, e.g. $\big\langle \hat{\rho}_1(z)\hat{\rho}_1(z')\big\rangle$ here. $\hat{\rho}_1(z)\hat{\rho}_1(z') = \hat{\psi}^\dagger(z)\hat{\psi}(z)\hat{\psi}^\dagger(z')\hat{\psi}(z')$ is Hermitian since it is sum of two Hermitian operators as follows

$$\hat{\rho}_1(z)\hat{\rho}_1(z') = \hat{\psi}^\dagger(z')\hat{\psi}^\dagger(z')\hat{\psi}(z)\hat{\psi}(z) + \delta(z - z')\hat{\rho}_1(z). \tag{7.93}$$

And later $\delta(z - z')$ comes from

$$\left[\hat{\psi}(z), \hat{\psi}^\dagger(z')\right] = \sum_{i=0}^{\infty} \psi_i^*(z')\psi_i(z) = \delta(z - z'). \tag{7.94}$$

However, if we truncate field operator as in (7.91) then we have

$$\left[\hat{\psi}(z), \hat{\psi}^\dagger(z')\right] = \psi_0^*(z')\psi_0(z) + \psi_1^*(z')\psi_1(z) \tag{7.95}$$

which is not necessarily real function, therefore we get not Hermitian part depending on $\psi_0(z)$ and $\psi_1(z)$. This actually yields non 0 imaginary part in $\langle\hat{\rho}_1(z)\hat{\rho}_1(z')\rangle$ thus clearly wrong result. By comparing two equations above, we can conclude that this imaginary part happens because just truncating field operator as in (7.91) did not perform two-mode approximation correctly.

Effective Hamiltonian from truncated field operator has limitation. Let us consider in general for which kind of operator $\hat{O}$ we can calculate by applying following truncation of field operator $\hat{\psi}(\mathbf{r})$

$$\hat{\psi}(\mathbf{r}) \tag{7.96}$$

to get $\langle\hat{O}\rangle$. To simplify an argument, we consider 'perfect' two-mode state $|\Psi\rangle$ such that

$$\langle\Psi|\,\hat{n}_i\,|\Psi\rangle = 0 \quad \text{for } i = M, M+1, \cdots \tag{7.97}$$

which leads to

$$\hat{a}_i\,|\Psi\rangle = 0, \;\; \langle\Psi|\,\hat{a}_i^{\dagger} = 0 \quad \text{for } i = M, M+1, \cdots . \tag{7.98}$$

This is also applied to mixed state of ensemble $\{|\Psi_n\rangle\}$ where

$$\langle\hat{n}_i\rangle = \sum_n P_n\,\langle\Psi_n|\,\hat{n}_i\,|\Psi_n\rangle = 0 \quad \text{for } i = M, M+1, \cdots . \tag{7.99}$$

with P_n which is probability in $n-th$ state $|\Psi_n\rangle$ and since

$$\langle\Psi_n|\,\hat{n}_i\,|\Psi_n\rangle \geq 0, \;\; P_n \geq 0 \tag{7.100}$$

we have

$$\hat{a}_i\,|\Psi_n\rangle = 0, \;\; \langle\Psi_n|\,\hat{a}_i^{\dagger} = 0 \quad \text{for } i = M, M+1, \cdots . \tag{7.101}$$

for all $|\Psi_n\rangle$ in ensemble.

In general, $\langle\hat{\psi}^{\dagger}(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\psi}^{\dagger}(\mathbf{r}')\hat{\psi}(\mathbf{r}')\rangle$ we can write as

$$\langle\hat{\psi}^{\dagger}(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\psi}^{\dagger}(\mathbf{r}')\hat{\psi}(\mathbf{r}')\rangle = \sum_{i,j,k,l=0}^{\infty} \psi_i^*(\mathbf{r})\psi_j(\mathbf{r})\psi_k^*(\mathbf{r}')\psi_l(\mathbf{r}')\langle\hat{a}_i^{\dagger}\hat{a}_j\hat{a}_k^{\dagger}\hat{a}_l\rangle \tag{7.102}$$

and immediately we see that if $i \geq M$ or $l \geq M$ then $\langle\hat{a}_i^{\dagger}\hat{a}_j\hat{a}_k^{\dagger}\hat{a}_l\rangle = 0$. Further, by giving normal ordering to $\langle \hat{a}_i^\dagger \hat{a}_j \hat{a}_k^\dagger \hat{a}_l \rangle$ we have

$$\langle \hat{a}_i^\dagger \hat{a}_j \hat{a}_k^\dagger \hat{a}_l \rangle = \langle \hat{a}_i^\dagger \hat{a}_k^\dagger \hat{a}_j \hat{a}_l \rangle + \langle \hat{a}_i^\dagger \left[\hat{a}_j, \hat{a}_k^\dagger \right] \hat{a}_l \rangle \tag{7.103}$$

therefore only $j = k$ survives for $j, k \geq M$ since $\hat{a}_j$ or $\hat{a}_k^\dagger$ will annihilate the state if $j \geq M$ or $k \geq M$. From this, we can also see that for normal ordered operator, it is enough to perform (A.1) to get M mode approximation because for normal ordered $\langle \hat{a}_i^\dagger \hat{a}_k^\dagger \hat{a}_j \hat{a}_l \rangle$ every i, j, k, l belongs to M mode configuration $(i, j, k, l < M)$ as long as (A.2) holds or is effectively suitable. Finally we get

$$\begin{aligned} &\langle \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}') \hat{\psi}(\mathbf{r}') \rangle = \\ &\sum_{i=0}^{M-1} \sum_{l=0}^{M-1} \left(\sum_{j=0}^{M-1} \sum_{k=0}^{M-1} + \sum_{j=M}^{\infty} \delta_{jk} \right) \psi_i^*(\mathbf{r}) \psi_j(\mathbf{r}) \psi_k^*(\mathbf{r}') \psi_l(\mathbf{r}') \langle \hat{a}_i^\dagger \hat{a}_j \hat{a}_k^\dagger \hat{a}_l \rangle. \end{aligned} \tag{7.104}$$

Now we know that second summation in bracket gives extra terms which we missed by directly applying (A.1) to $\langle \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}') \hat{\psi}(\mathbf{r}') \rangle$. In conclusion, to avoid such trouble we should perform truncation of filed operator after we express target operator $\hat{O}$ to be calculated as sum of normal ordered operators which comes from commuting every annihilation operators at the left of creation operators to the right of creation operators.

For $M = 2$ case in one dimension (z), we can perform (7.91) to right hand side of equality below

$$\hat{\rho}_1(z) \hat{\rho}_1(z') = \hat{\psi}^\dagger(z') \hat{\psi}^\dagger(z') \hat{\psi}(z) \hat{\psi}(z) + \delta(z - z') \hat{\rho}_1(z). \tag{7.105}$$

So when $z \neq z'$ we have $\langle \hat{\rho}_1(z) \hat{\rho}_1(z') \rangle = \langle \hat{\psi}^\dagger(z') \hat{\psi}^\dagger(z') \hat{\psi}(z) \hat{\psi}(z) \rangle$ and this is written as

$$
\begin{aligned}
&\big\langle \hat\rho_1(z)\hat\rho_1(z')\big\rangle = \big\langle \hat\psi^\dagger(z)\hat\psi^\dagger(z')\hat\psi(z')\hat\psi(z)\big\rangle = |\psi_0(z)|^2|\psi_0(z')|^2\big\langle \hat a_0^\dagger\hat a_0^\dagger\hat a_0\hat a_0\big\rangle\\
&+|\psi_1(z)|^2|\psi_1(z')|^2\big\langle \hat a_1^\dagger\hat a_1^\dagger\hat a_1\hat a_1\big\rangle\\
&+\left(|\psi_0(z)|^2|\psi_1(z')|^2+|\psi_1(z)|^2|\psi_0(z')|^2\right)\big\langle \hat a_0^\dagger\hat a_1^\dagger\hat a_1\hat a_0\big\rangle\\
&+\left(|\psi_0(z)|^2\psi_1^*(z')\psi_0(z')+|\psi_0(z')|^2\psi_1^*(z)\psi_0(z)\right)\big\langle \hat a_1^\dagger\hat a_0^\dagger\hat a_0\hat a_0\big\rangle\\
&+\left(|\psi_0(z)|^2\psi_0^*(z')\psi_1(z')+|\psi_0(z')|^2\psi_0^*(z)\psi_1(z)\right)\big\langle \hat a_0^\dagger\hat a_0^\dagger\hat a_0\hat a_1\big\rangle\\
&+\left(|\psi_1(z)|^2\psi_0^*(z')\psi_1(z')+|\psi_1(z')|^2\psi_0^*(z)\psi_1(z)\right)\big\langle \hat a_0^\dagger\hat a_1^\dagger\hat a_1\hat a_1\big\rangle\\
&+\left(|\psi_1(z)|^2\psi_1^*(z')\psi_0(z')+|\psi_1(z')|^2\psi_1^*(z)\psi_0(z)\right)\big\langle \hat a_1^\dagger\hat a_1^\dagger\hat a_1\hat a_0\big\rangle\\
&+\left(\psi_0^*(z)\psi_1^*(z')\psi_0(z')\psi_1(z)+\text{h.c.}\right)\big\langle \hat a_0^\dagger\hat a_1^\dagger\hat a_1\hat a_0\big\rangle\\
&+\Big(\psi_0^*(z)\psi_0^*(z')\psi_1(z')\psi_1(z)\big\langle \hat a_0^\dagger\hat a_0^\dagger\hat a_1\hat a_1\big\rangle+\text{h.c.}\Big)
\end{aligned}
\tag{7.106}
$$

and $\Delta\rho_2(z,z')$ is (when $z\neq z'$)

$$
\begin{aligned}
&\Delta\rho_2(z,z') = \big\langle \hat\psi^\dagger(z)\hat\psi^\dagger(z')\hat\psi(z')\hat\psi(z)\big\rangle - \big\langle\hat\rho_1(z)\big\rangle\big\langle\hat\rho_1(z')\big\rangle\\
&\Big(|\psi_0(z)|^2\big\langle \hat a_0^\dagger\hat a_0\big\rangle+|\psi_1(z)|^2\big\langle \hat a_1^\dagger\hat a_1\big\rangle+\psi_0^*(z)\psi_1(z)\big\langle \hat a_0^\dagger\hat a_1\big\rangle+\psi_1^*(z)\psi_0(z)\big\langle \hat a_1^\dagger\hat a_0\big\rangle\Big)\times\\
&\Big(|\psi_0(z')|^2\big\langle \hat a_0^\dagger\hat a_0\big\rangle+|\psi_1(z')|^2\big\langle \hat a_1^\dagger\hat a_1\big\rangle+\psi_0^*(z')\psi_1(z')\big\langle \hat a_0^\dagger\hat a_1\big\rangle+\psi_1^*(z')\psi_0(z')\big\langle \hat a_1^\dagger\hat a_0\big\rangle\Big)\\
&=|\psi_0(z)|^2|\psi_0(z')|^2\Big(\big\langle \hat a_0^\dagger\hat a_0^\dagger\hat a_0\hat a_0\big\rangle-\big\langle \hat a_0^\dagger\hat a_0\big\rangle\big\langle \hat a_0^\dagger\hat a_0\big\rangle\Big)\\
&+|\psi_1(z)|^2|\psi_1(z')|^2\Big(\big\langle \hat a_1^\dagger\hat a_1^\dagger\hat a_1\hat a_1\big\rangle-\big\langle \hat a_1^\dagger\hat a_1\big\rangle\big\langle \hat a_1^\dagger\hat a_1\big\rangle\Big)\\
&+\left(|\psi_0(z)|^2|\psi_1(z')|^2+|\psi_1(z)|^2|\psi_0(z')|^2\right)\Big(\big\langle \hat a_0^\dagger\hat a_1^\dagger\hat a_1\hat a_0\big\rangle-\big\langle \hat a_0^\dagger\hat a_0\big\rangle\big\langle \hat a_1^\dagger\hat a_1\big\rangle\Big)\\
&+\left(|\psi_0(z)|^2\psi_1^*(z')\psi_0(z')+|\psi_0(z')|^2\psi_1^*(z)\psi_0(z)\right)\Big(\big\langle \hat a_1^\dagger\hat a_0^\dagger\hat a_0\hat a_0\big\rangle-\big\langle \hat a_1^\dagger\hat a_0\big\rangle\big\langle \hat a_0^\dagger\hat a_0\big\rangle\Big)\\
&+\left(|\psi_0(z)|^2\psi_0^*(z')\psi_1(z')+|\psi_0(z')|^2\psi_0^*(z)\psi_1(z)\right)\Big(\big\langle \hat a_0^\dagger\hat a_0^\dagger\hat a_0\hat a_1\big\rangle-\big\langle \hat a_0^\dagger\hat a_0\big\rangle\big\langle \hat a_0^\dagger\hat a_1\big\rangle\Big)\\
&+\left(|\psi_1(z)|^2\psi_0^*(z')\psi_1(z')+|\psi_1(z')|^2\psi_0^*(z)\psi_1(z)\right)\Big(\big\langle \hat a_0^\dagger\hat a_1^\dagger\hat a_1\hat a_1\big\rangle-\big\langle \hat a_0^\dagger\hat a_1\big\rangle\big\langle \hat a_1^\dagger\hat a_1\big\rangle\Big)\\
&+\left(|\psi_1(z)|^2\psi_1^*(z')\psi_0(z')+|\psi_1(z')|^2\psi_1^*(z)\psi_0(z)\right)\Big(\big\langle \hat a_1^\dagger\hat a_1^\dagger\hat a_1\hat a_0\big\rangle-\big\langle \hat a_1^\dagger\hat a_1\big\rangle\big\langle \hat a_1^\dagger\hat a_0\big\rangle\Big)\\
&+\left(\psi_0^*(z)\psi_1^*(z')\psi_0(z')\psi_1(z)+\text{h.c.}\right)\Big(\big\langle \hat a_0^\dagger\hat a_1^\dagger\hat a_1\hat a_0\big\rangle-\big\langle \hat a_0^\dagger\hat a_1\big\rangle\big\langle \hat a_1^\dagger\hat a_0\big\rangle\Big)\\
&+\Big(\psi_0^*(z)\psi_0^*(z')\psi_1(z')\psi_1(z)\Big(\big\langle \hat a_0^\dagger\hat a_0^\dagger\hat a_1\hat a_1\big\rangle-\big\langle \hat a_0^\dagger\hat a_1\big\rangle\big\langle \hat a_0^\dagger\hat a_1\big\rangle\Big)+\text{h.c.}\Big).
\end{aligned}
\tag{7.107}
$$

Now we consider orbitals $\psi_0(z),\psi_1(z)\in\mathbb{R}$ before TOF. Then, we get following (again) very long expression

$$
\begin{aligned}
&\Delta\rho_2(z,z') = |\psi_0(z)|^2|\psi_0(z')|^2 \left(\langle \hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_0\hat{a}_0\rangle - \langle \hat{a}_0^\dagger\hat{a}_0\rangle\langle \hat{a}_0^\dagger\hat{a}_0\rangle\right)\\
&+|\psi_1(z)|^2|\psi_1(z')|^2 \left(\langle \hat{a}_1^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_1\rangle - \langle \hat{a}_1^\dagger\hat{a}_1\rangle\langle \hat{a}_1^\dagger\hat{a}_1\rangle\right)\\
&+\left(|\psi_0(z)|^2|\psi_1(z')|^2 + |\psi_1(z)|^2|\psi_0(z')|^2\right)\left(\langle \hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0\rangle - \langle \hat{a}_0^\dagger\hat{a}_0\rangle\langle \hat{a}_1^\dagger\hat{a}_1\rangle\right)\\
&+\left(|\psi_0(z)|^2\psi_0(z')\psi_1(z') + |\psi_0(z')|^2\psi_0(z)\psi_1(z)\right)\\
&\quad\times\left[\left(\langle \hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_0\hat{a}_1\rangle - \langle \hat{a}_0^\dagger\hat{a}_0\rangle\langle \hat{a}_0^\dagger\hat{a}_1\rangle\right) + \text{ h.c.}\right]\\
&+\left(|\psi_1(z)|^2\psi_0(z')\psi_1(z') + |\psi_1(z')|^2\psi_0(z)\psi_1(z)\right)\\
&\quad\times\left[\left(\langle \hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_1\rangle - \langle \hat{a}_0^\dagger\hat{a}_1\rangle\langle \hat{a}_1^\dagger\hat{a}_1\rangle\right) + \text{ h.c.}\right]\\
&+\psi_0(z)\psi_1(z')\psi_0(z')\psi_1(z)\\
&\quad\times\left[2\langle \hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0\rangle + \langle \hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_1\hat{a}_1\rangle + \langle \hat{a}_1^\dagger\hat{a}_1^\dagger\hat{a}_0\hat{a}_0\rangle - \left(\langle \hat{a}_0^\dagger\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_0\rangle\right)^2\right].
\end{aligned}
\tag{7.108}
$$

It is noted that above expression is also available when $\psi_0^*(z)\psi_1(z) \in \mathbb{R}$. Looking at square bracket of the last line of above (7.108), we find that

$$
\begin{aligned}
&2\langle \hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0\rangle + \langle \hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_1\hat{a}_1\rangle + \langle \hat{a}_1^\dagger\hat{a}_1^\dagger\hat{a}_0\hat{a}_0\rangle - \left(\langle \hat{a}_0^\dagger\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_0\rangle\right)^2\\
&= \left[\langle \left(\hat{a}_0^\dagger\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_0\right)^2 \rangle - \left(\langle \hat{a}_0^\dagger\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_0\rangle\right)^2\right] - \langle \hat{a}_0^\dagger\hat{a}_0\rangle - \langle \hat{a}_1^\dagger\hat{a}_1\rangle
\end{aligned}
\tag{7.109}
$$

therefore this term is deeply related to fluctuation of quadrature $(\hat{b} + \hat{b}^\dagger)/\sqrt{2}$ except last two terms from commutator.

After TOF, we have $\psi_0(z,t) = \psi_0(-z,t), \psi_1(z,t) = -\psi_1(-z,t)$ with

$$
i\psi_1(z,t) \equiv \bar{\psi}_1(z,t), \quad |\bar{\psi}_1(z,t)|^2 = |\psi_1(z,t)|^2 \tag{7.110}
$$

and $\psi_0^*(z,t)\bar{\psi}_1(z,t) \in \mathbb{R}$ where $\psi_0(z,t)$ and $\psi_1(z,t)$ share a phase factor which depend on position and time keeping $\langle \hat{a}_i^\dagger\hat{a}_j\rangle, \langle \hat{a}_i^\dagger\hat{a}_j^\dagger\hat{a}_k\hat{a}_l\rangle$ $(i,j,k,l = 0,1)$ invariant over time evolution in weakly interacting limit of TOF. Then we have

$$
\begin{aligned}
&\Delta\rho_2(z,z',t) = |\psi_0(z,t)|^2|\psi_0(z',t)|^2 \left(\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0\rangle - \langle \hat{a}_0^\dagger \hat{a}_0\rangle\langle \hat{a}_0^\dagger \hat{a}_0\rangle\right) \\
&+|\psi_1(z,t)|^2|\psi_1(z',t)|^2 \left(\langle \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1\rangle - \langle \hat{a}_1^\dagger \hat{a}_1\rangle\langle \hat{a}_1^\dagger \hat{a}_1\rangle\right) \\
&+ \left(|\psi_0(z,t)|^2|\psi_1(z',t)|^2 + |\psi_1(z,t)|^2|\psi_0(z',t)|^2\right) \left(\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0\rangle - \langle \hat{a}_0^\dagger \hat{a}_0\rangle\langle \hat{a}_1^\dagger \hat{a}_1\rangle\right) \\
&+ \left(|\psi_0(z,t)|^2\psi_0^*(z',t)\bar{\psi}_1(z',t) + |\psi_0(z',t)|^2\psi_0^*(z,t)\bar{\psi}_1(z,t)\right) \\
&\quad\times \left[-i\left(\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_1\rangle - \langle \hat{a}_0^\dagger \hat{a}_0\rangle\langle \hat{a}_0^\dagger \hat{a}_1\rangle\right) + \text{ h.c.}\right] \\
&+ \left(|\psi_1(z,t)|^2\psi_0^*(z',t)\bar{\psi}_1(z',t) + |\psi_1(z',t)|^2\psi_0^*(z,t)\bar{\psi}_1(z,t)\right) \\
&\quad\times \left[-i\left(\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1\rangle - \langle \hat{a}_0^\dagger \hat{a}_1\rangle\langle \hat{a}_1^\dagger \hat{a}_1\rangle\right) + \text{ h.c.}\right] \\
&\quad+ \psi_0^*(z,t)\bar{\psi}_1(z,t)\psi_0^*(z',t)\bar{\psi}_1(z',t) \\
&\quad\times \left[2\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0\rangle - \langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1\rangle - \langle \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_0 \hat{a}_0\rangle + \left(\langle \hat{a}_0^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_0\rangle\right)^2\right].
\end{aligned}
\tag{7.111}
$$

This time from square bracket of the last line of above (7.111), we find that

$$
\begin{aligned}
&- \langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1\rangle - \langle \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_0 \hat{a}_0\rangle + 2\langle \hat{a}_0^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_0\rangle + \left(\langle \hat{a}_0^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_0\rangle\right)^2 \\
&= -\left[\left\langle \left(\hat{a}_0^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_0\right)^2 \right\rangle - \left(\langle \hat{a}_0^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_0\rangle\right)^2\right] - \langle \hat{a}_0^\dagger \hat{a}_0\rangle - \langle \hat{a}_1^\dagger \hat{a}_1\rangle
\end{aligned}
\tag{7.112}
$$

this term is deeply related to fluctuation of quadrature $(\hat{b} - \hat{b}^\dagger)/\sqrt{2}i$ except last two terms from commutator.

We'd like generalize TOF evolution a bit more by considering

$$
e^{-i\varphi}\psi_1(z,t) \equiv \bar{\psi}_1(z,t), \quad |\bar{\psi}_1(z,t)|^2 = |\psi_1(z,t)|^2 \tag{7.113}
$$

and $\psi_0^*(z,t)\bar{\psi}_1(z,t) \in \mathbb{R}$ which includes above TOF case from $\varphi = 3\pi/2 = -\pi/2$. This happens when phase factors of $\psi_0(z,t)$ and $\psi_1(z,t)$ is shared (of the same) which depends on position and time up to constant phase at the same time t. Then we can find φ such that $\psi_0^*(z,t)\bar{\psi}_1(z,t) \in \mathbb{R}$ where rotating phase of $\psi_1(z,t)$ by $-\varphi$ gives $\bar{\psi}_1(z,t)$. We would like to note that this value of φ could be both φ or $\varphi + \pi$ since $e^{i\pi}$ times real number is again real number.

Then we can write $\Delta\rho_2(z,z',t)$ as

$$
\begin{aligned}
&\Delta\rho_2(z,z',t) = |\psi_0(z,t)|^2|\psi_0(z',t)|^2\left(\langle\hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_0\hat{a}_0\rangle - \langle\hat{a}_0^\dagger\hat{a}_0\rangle\langle\hat{a}_0^\dagger\hat{a}_0\rangle\right)\\
&+|\psi_1(z,t)|^2|\psi_1(z',t)|^2\left(\langle\hat{a}_1^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_1\rangle - \langle\hat{a}_1^\dagger\hat{a}_1\rangle\langle\hat{a}_1^\dagger\hat{a}_1\rangle\right)\\
&+\left(|\psi_0(z,t)|^2|\psi_1(z',t)|^2+|\psi_1(z,t)|^2|\psi_0(z',t)|^2\right)\left(\langle\hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0\rangle - \langle\hat{a}_0^\dagger\hat{a}_0\rangle\langle\hat{a}_1^\dagger\hat{a}_1\rangle\right)\\
&+\left(|\psi_0(z,t)|^2\psi_0^*(z',t)\bar{\psi}_1(z',t)+|\psi_0(z',t)|^2\psi_0^*(z,t)\bar{\psi}_1(z,t)\right)\\
&\quad\times\left[e^{i\varphi}\left(\langle\hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_0\hat{a}_1\rangle - \langle\hat{a}_0^\dagger\hat{a}_0\rangle\langle\hat{a}_0^\dagger\hat{a}_1\rangle\right)+\text{ h.c.}\right]\\
&+\left(|\psi_1(z,t)|^2\psi_0^*(z',t)\bar{\psi}_1(z',t)+|\psi_1(z',t)|^2\psi_0^*(z,t)\bar{\psi}_1(z,t)\right)\\
&\quad\times\left[e^{i\varphi}\left(\langle\hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_1\rangle - \langle\hat{a}_0^\dagger\hat{a}_1\rangle\langle\hat{a}_1^\dagger\hat{a}_1\rangle\right)+\text{ h.c.}\right]\\
&+\psi_0^*(z,t)\bar{\psi}_1(z,t)\psi_0^*(z',t)\bar{\psi}_1(z',t)\Big[2\langle\hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0\rangle + e^{2i\varphi}\langle\hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_1\hat{a}_1\rangle\\
&\quad+e^{-2i\varphi}\langle\hat{a}_1^\dagger\hat{a}_1^\dagger\hat{a}_0\hat{a}_0\rangle - \left(\langle e^{i\varphi}\hat{a}_0^\dagger\hat{a}_1+e^{-i\varphi}\hat{a}_1^\dagger\hat{a}_0\rangle\right)^2\Big].
\end{aligned}
\tag{7.114}
$$

Square bracket of the last line of above expression can be written as

$$
\begin{aligned}
&2\langle\hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0\rangle + e^{2i\varphi}\langle\hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_1\hat{a}_1\rangle + e^{-2i\varphi}\langle\hat{a}_1^\dagger\hat{a}_1^\dagger\hat{a}_0\hat{a}_0\rangle - \left(\langle e^{i\varphi}\hat{a}_0^\dagger\hat{a}_1+e^{-i\varphi}\hat{a}_1^\dagger\hat{a}_0\rangle\right)^2\\
&=\left[\langle\left(e^{i\varphi}\hat{a}_0^\dagger\hat{a}_1+e^{-i\varphi}\hat{a}_1^\dagger\hat{a}_0\right)^2\rangle - \left(\langle e^{i\varphi}\hat{a}_0^\dagger\hat{a}_1+e^{-i\varphi}\hat{a}_1^\dagger\hat{a}_0\rangle\right)^2\right] - \langle\hat{a}_0^\dagger\hat{a}_0\rangle - \langle\hat{a}_1^\dagger\hat{a}_1\rangle.
\end{aligned}
\tag{7.115}
$$

Except commutator part, we can express terms in square bracket as fluctuation of linear combination of quadratures as follows

$$
\begin{aligned}
&\langle\left(\cos\varphi\left(\hat{a}_0^\dagger\hat{a}_1+\hat{a}_1^\dagger\hat{a}_0\right)+i\sin\varphi\left(\hat{a}_0^\dagger\hat{a}_1-\hat{a}_1^\dagger\hat{a}_0\right)\right)^2\rangle\\
&\qquad-\left(\langle\cos\varphi\left(\hat{a}_0^\dagger\hat{a}_1+\hat{a}_1^\dagger\hat{a}_0\right)+i\sin\varphi\left(\hat{a}_0^\dagger\hat{a}_1-\hat{a}_1^\dagger\hat{a}_0\right)\rangle\right)^2\\
&=2\left[\langle\left(\cos\varphi\frac{\hat{a}_0^\dagger\hat{a}_1+\hat{a}_1^\dagger\hat{a}_0}{\sqrt{2}}-\sin\varphi\frac{\hat{a}_0^\dagger\hat{a}_1-\hat{a}_1^\dagger\hat{a}_0}{\sqrt{2}i}\right)^2\rangle\right.\\
&\qquad\left.-\left(\langle\cos\varphi\frac{\hat{a}_0^\dagger\hat{a}_1+\hat{a}_1^\dagger\hat{a}_0}{\sqrt{2}}-\sin\varphi\frac{\hat{a}_0^\dagger\hat{a}_1-\hat{a}_1^\dagger\hat{a}_0}{\sqrt{2}i}\rangle\right)^2\right].
\end{aligned}
\tag{7.116}
$$

Further, we can take $\bar{\psi}_1(z,t)$ as new $\psi_1(z,t)$. Then we have $\psi_0^*(z,t)\psi_1(z,t)\in\mathbb{R}$, instead we change the many-body state $|\Psi\rangle$ as follows

$$|\Psi\rangle = \sum_{l=0}^{N} C_l \left|N-l,l\right\rangle \rightarrow |\Psi_\varphi\rangle = \sum_{l=0}^{N} C_l e^{il\varphi} \left|N-l,l\right\rangle \tag{7.117}$$

which allows us to use an expression (7.111) as

$$\begin{aligned}
&\Delta\rho_2(z,z',t) = |\psi_0(z,t)|^2|\psi_0(z',t)|^2 \left(\left\langle \hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_0\hat{a}_0\right\rangle_\varphi - \left\langle \hat{a}_0^\dagger\hat{a}_0\right\rangle_\varphi \left\langle \hat{a}_0^\dagger\hat{a}_0\right\rangle_\varphi\right) \\
&+|\psi_1(z,t)|^2|\psi_1(z',t)|^2 \left(\left\langle \hat{a}_1^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_1\right\rangle_\varphi - \left\langle \hat{a}_1^\dagger\hat{a}_1\right\rangle_\varphi \left\langle \hat{a}_1^\dagger\hat{a}_1\right\rangle_\varphi\right) \\
&+\left(|\psi_0(z,t)|^2|\psi_1(z',t)|^2 + |\psi_1(z,t)|^2|\psi_0(z',t)|^2\right) \left(\left\langle \hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0\right\rangle_\varphi - \left\langle \hat{a}_0^\dagger\hat{a}_0\right\rangle_\varphi \left\langle \hat{a}_1^\dagger\hat{a}_1\right\rangle_\varphi\right) \\
&+\left(|\psi_0(z,t)|^2\psi_0^*(z',t)\psi_1(z',t) + |\psi_0(z',t)|^2\psi_0^*(z,t)\psi_1(z,t)\right) \\
&\quad\times \left[\left(\left\langle \hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_0\hat{a}_1\right\rangle_\varphi - \left\langle \hat{a}_0^\dagger\hat{a}_0\right\rangle_\varphi \left\langle \hat{a}_0^\dagger\hat{a}_1\right\rangle_\varphi\right) + \text{ h.c.}\right] \\
&+\left(|\psi_1(z,t)|^2\psi_0^*(z',t)\psi_1(z',t) + |\psi_1(z',t)|^2\psi_0^*(z,t)\psi_1(z,t)\right) \\
&\quad\times \left[\left(\left\langle \hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_1\right\rangle_\varphi - \left\langle \hat{a}_0^\dagger\hat{a}_1\right\rangle_\varphi \left\langle \hat{a}_1^\dagger\hat{a}_1\right\rangle_\varphi\right) + \text{ h.c.}\right] \\
&+\psi_0^*(z,t)\bar{\psi}_1(z,t)\psi_0^*(z',t)\bar{\psi}_1(z',t) \\
&\quad\times \left[2\left\langle \hat{a}_0^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_0\right\rangle_\varphi + \left\langle \hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_1\hat{a}_1\right\rangle_\varphi + \left\langle \hat{a}_1^\dagger\hat{a}_1^\dagger\hat{a}_0\hat{a}_0\right\rangle_\varphi - \left(\left\langle \hat{a}_0^\dagger\hat{a}_1 + \hat{a}_1^\dagger\hat{a}_0\right\rangle_\varphi\right)^2\right].
\end{aligned} \tag{7.118}$$

with $\varphi = -\pi/2 = 3\pi/2$ and also (7.108) with $\varphi = 0, t = 0$.

Here $\langle\rangle_\varphi$ is an expectation value over $|\Psi_\varphi\rangle$ related to $\langle\rangle$ by

$$\left\langle (\hat{a}_0^\dagger)^a(\hat{a}_1^\dagger)^b(\hat{a}_0)^c(\hat{a}_1)^d\right\rangle_\varphi = \left\langle (\hat{a}_0^\dagger)^a(\hat{a}_1^\dagger)^b(\hat{a}_0)^c(\hat{a}_1)^d\right\rangle e^{-i(b-d)\varphi} \tag{7.119}$$

therefore $\hat{a}^\dagger$ yields $e^{-i\varphi}$ and $\hat{a}$ yields $e^{i\varphi}$. One can immediately check that an above expression of $\Delta\rho_2(z,z')$ produces the same result as (7.114). This rotation of state in (7.117) corresponds to rotation of ϕ_β for approximate coherent state $|\beta\rangle$ as follows.

$$|\beta\rangle \rightarrow \left|\beta e^{i\varphi}\right\rangle \tag{7.120}$$

and for following general NPC state which can be expressed as a superposition of $|\beta\rangle$ and $|-\beta\rangle$ we get

$$\begin{aligned}&\frac{1}{\sqrt{1+r^2+2r\cos\theta e^{-2|\beta|^2}}}\left(|\beta\rangle + re^{i\theta}\,|-\beta\rangle\right)\\&\qquad\to \frac{1}{\sqrt{1+r^2+2r\cos\theta e^{-2|\beta|^2}}}\left(\left|\beta e^{i\varphi}\right\rangle + re^{i\theta}\left|-\beta e^{i\varphi}\right\rangle\right).\end{aligned} \tag{7.121}$$

Since TOF evolution is equivalent to $\varphi = 3\pi/2 = -\pi/2$ rotation of state defined in (7.117), we can summarize as follows: For NPC state consisting of ψ_0, psi_1 with even-odd parity, we can express $\Delta\rho_2(z,z')$ as in (7.118) both for before TOF ($\varphi = 0$) and for after TOF ($\varphi = 3\pi/2$). And in both cases we have terms related to quadrature fluctuation rather directly which is evaluated against rotated $|\Psi_\varphi\rangle$.

Using results for general NPC states in chapter 5.2.1, (7.114) becomes

$$\begin{aligned}&\Delta\rho_2(z,z',t) \simeq |\psi_0(z,t)|^2|\psi_0(z',t)|^2\left(\sigma^2-(N-l_0)\right)\\&+|\psi_1(z,t)|^2|\psi_1(z',t)|^2\left(\sigma^2-l_0\right)+\left(|\psi_0(z,t)|^2|\psi_1(z',t)|^2+z\leftrightarrow z'\right)\sigma^2\\&+\left((|\psi_0(z,t)|^2+|\psi_1(z,t)|^2)\psi_0^*(z',t)\bar{\psi}_1(z',t)+z\leftrightarrow z'\right)\left(4\sigma^2|c|^2u\sin\varphi\right)\\&+\psi_0^*(z,t)\bar{\psi}_1(z,t)\psi_0^*(z',t)\bar{\psi}_1(z',t)\\&\times\left[4(N-l_0)l_0\sin^2\varphi\left(1-(2|c|^2u\sin\theta_k)^2\right)+2\sqrt{(N-l_0)l_0}\cos 2\varphi-4\sigma^2\sin^2\varphi\right].\end{aligned} \tag{7.122}$$

(7.122) can be written as except $\mathcal{O}(N-l_0)+\mathcal{O}(\sigma^2)$

$$\begin{aligned}&\Delta\rho_2(z,z',t)\simeq\\&\quad 4\psi_0^*(z,t)\bar{\psi}_1(z,t)\psi_0^*(z',t)\bar{\psi}_1(z',t)(N-l_0)l_0\sin^2\varphi\left(1-(2|c|^2u\sin\theta_k)^2\right)\end{aligned} \tag{7.123}$$

when l_0 and σ^2 are $\mathcal{O}(N-l_0)$ (or smaller). If $N-l_0, l_0 \gg 1$ so

$$(N-l_0)l_0 \gg N-l_0, l_0 \tag{7.124}$$

then unless

$$|\psi_0(z,t)|^2|\psi_0(z',t)|^2,\ |\psi_1(z,t)|^2|\psi_1(z',t)|^2,\ (|\psi_0(z,t)|^2|\psi_1(z',t)|^2 z \leftrightarrow z')$$
$$\gg \psi_0^*(z,t)\bar{\psi}_1(z,t)\psi_0^*(z',t)\bar{\psi}_1(z',t) \tag{7.125}$$

$\Delta\rho_2(z,z',t)$ can be expressed simply as

$$\Delta\rho_2(z,z',t) \simeq$$
$$4\psi_0^*(z,t)\bar{\psi}_1(z,t)\psi_0^*(z',t)\bar{\psi}_1(z',t)(N-l_0)l_0\sin^2\varphi\left(1-(2|c|^2u\sin\theta_k)^2\right). \tag{7.126}$$

It is noted that this result holds for general NPC state which satisfies (7.124) and (7.125) irrelevant of ψ_0 or ψ_1 as long as general φ is considered (though even odd parity of ψ_0 and ψ_1 leads to NPC state and at the same time it is one and only candidate now). Actually φ and $\bar{\psi}_1$ fixes rather arbitrary relative phase between ψ_0 and ψ_1 from condition $\psi_0^*(z,t)\bar{\psi}_1(z,t) \in \mathbb{R}$.

Now we are going to plot $\Delta\rho_2(z,z',t)$ from (7.114) for each before TOF ($\varphi = 0$) and after TOF given $\psi_0(z), \psi_1(z)$ which are each ground state and 1st excited state (we have then $\varphi = 3\pi/2 = -\pi/2$ from TOF) of simple harmonic oscillator. We calculate $\Delta\rho_2(z,z',t)$ for following superposition state in (7.63)

$$\frac{1}{\sqrt{1+r^2+2r\cos\theta e^{-2|\beta|^2}}}\left(|\beta\rangle + re^{i\theta}|-\beta\rangle\right) \tag{7.127}$$

for various value of $|\beta|^2$ and r.

To interpret result based on (7.126), here we again briefly write a relation between $2|c|^2u\sin\theta_k$ and r,θ since only $2|c|^2u\sin\theta_k$ is the term which is affected by different r,θ. a,b used here are those in (5.66) but we can consider c,u,θ_k as of the same used in (7.24) as long as NPC state is well described by (7.63) as intended so with

$$|\text{Even}\rangle \simeq \frac{1}{\sqrt{2(1+e^{-2|\beta|^2})}}(|\beta\rangle + |-\beta\rangle),\ \ |\text{Odd}\rangle \simeq \frac{1}{i\sqrt{2(1-e^{-2|\beta|^2})}}(|\beta\rangle - |-\beta\rangle) \tag{7.128}$$

where general NPC state $|NPC\rangle$ can be described by $c(|\text{Even}\rangle + ue^{i\theta_k}|\text{Odd}\rangle)$ with normalization condition $|c|^2 = 1/(1+u^2)$.

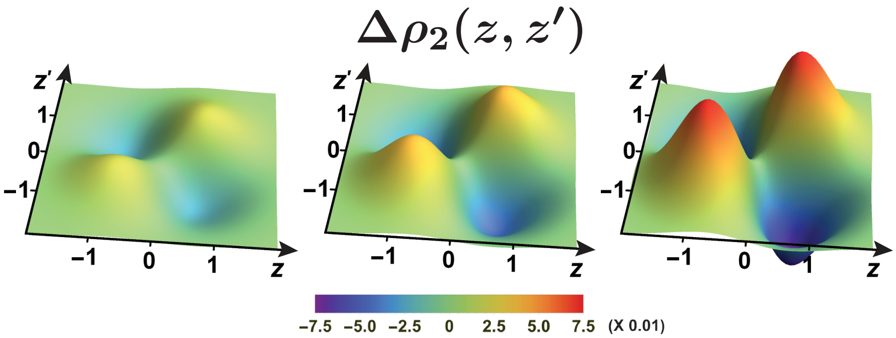

Figure 7.3: $\Delta\rho_2(z,z',t)$ after TOF scaled by N^2/Z^2 for $r=1$, and Z is TOF scale of z,z' [36]. Degree of fragmentation defined in (2.1) increases as $\mathcal{F}=0.1,0.2,0.4$ ($N=100$, $|\beta|^2=5,10,20$)

Then we have $2|c|^2 u\sin\theta_k = 2u/(1+u^2)\sin\theta_k$ as

$$\frac{1-r^2}{1+r^2} = \frac{2u\lambda_\beta}{1+u^2\lambda_\beta^2}\sin\theta_k = \left(\frac{2u}{1+u^2}\sin\theta_k\right)\frac{1+u^2}{\lambda_\beta(\lambda_\beta^{-2}+u^2)} \tag{7.129}$$

and for $\lambda_\beta = \sqrt{(1+e^{-2|\beta|^2})/(1-e^{-2|\beta|^2})} \to 1$ limit ($e^{-2|\beta|^2}\to 0$) it becomes

$$\frac{1-r^2}{1+r^2} = \frac{2u}{1+u^2}\sin\theta_k = 2|c|^2 u\sin\theta_k \tag{7.130}$$

which does not depends on θ. Thus we see that as overlap between $|\beta\rangle$ and $|-\beta\rangle$ gets smaller then an effect of θ on $\Delta\rho_2(z,z')$ becomes negligible. In this limit (7.126) in terms of r is given as

$$\Delta\rho_2(z,z',t) \simeq 4\psi_0^*(z,t)\bar\psi_1(z,t)\psi_0^*(z',t)\bar\psi_1(z',t)(N-l_0)l_0\sin^2\varphi\frac{4r^2}{(1+r^2)^2} \tag{7.131}$$

and from $\frac{4r^2}{(1+r^2)^2} = 4/(r+\frac{1}{r})^2$ we see that as r approaches to 1, $\Delta\rho_2(z,z',t)$ is maximized for given β. Fig.7.3 is plot of $\Delta\rho_2(z,z',t)$ after TOF for various values of $|\beta|^2$ for $N=100$.

Chapter 8

Conclusion

In this thesis, fragmentation in many-body system was investigated for dilute cold atomic gas in a single trap geometry.

Assuming additional macroscopic orbital $\psi_1(z)$ to have odd parity, as interaction strength increases fragmented state has lower energy than BEC with limited variational calculation. There exist almost degenerated two types of fragmented state $|\text{Even}\rangle$ and $|\text{Odd}\rangle$ with gap $\sim \epsilon_0 - \epsilon_1$. However, both states are fragile against small tunneling term therefore one of $|\text{Even}\rangle \pm |\text{Odd}\rangle$ is preferred. These fragmented states are stable against perturbation[20].

Due to explosive degree of freedom in both orbitals and C_l distribution, it is impossible to solve two-mode Hamiltonian by full variational calculation. Instead of theoretical proof, finding measurable distinction from BEC for fragmented state is realistic way to prove an existence of fragmented state. To find evidence of fragmented state in a single trap, spatial coherence $\rho_1(z, z')$ and density-density correlation $\rho_2(z, z')$ were investigated for BEC and fragmented state in a single trap. $\rho_2(z, z')$ showed strong fluctuation comparable to correlation itself as degree of fragmentation $\mathcal{F}$ increases. Density-density correlation of double well fragmented state with HBT correlation was reviewed to compare with single-trap fragmented state. In single-trap fragmented state, there was no significant sign of HBT correlation displayed in $\rho_2(z, z')$.

Two major factors related to strong fluctuation were negative pair coherence $\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_1 \hat{a}_1 \rangle$ and $\pi/2$ rotation of relative phase between two-mods ψ_0 and ψ_1 during TOF. Later is from the fact that TOF visualizes initial momentum distribution as density distribution after TOF. And negative pair coherence, which is also related to positive A_3 -a source of fragmentation with $\mathrm{sgn}(C_l C_{l+2}) = -1$-, is very heart of fragmented state of single-trap.

Phase state formalism, as shown in independent two BECs case, is powerful formalism to analyze two-mode state when correlation function of certain two-mode system can be calculated from diagonal expression in integration over one phase ϕ. Condition to be applied fragmented state was found in rather brute way, and was confirmed to use diagonal expression in phase state basis. Further, condition on $|C_\phi|$ for fragmented state $\int d\phi |C_\phi|^2 e^{i\phi} = 0$ infers that it is likely to have peaks at certain value of ϕ unless $|C_\phi|$ is constant.

Approximate coherent state $|\beta\rangle$ was established for two-mode system with annihilation, creation operator $\hat{b}, \hat{b}^\dagger$. $|\beta\rangle, \hat{b}, \hat{b}^\dagger$ behaves really like actual coherent state and bosonic operators $|\alpha\rangle, \hat{a}, \hat{a}^\dagger$ when N is larger than few tens and value of $|\beta|^2 < N/2$. Including $|\beta'\rangle, \hat{b'}, \hat{b'}^\dagger$ working for $|\beta|^2 > N/2$, approximate coherent state and corresponding bosonic operators work quite well for two-mode system if $N > 100$. Similarities between phase state and $|\beta\rangle$ infers relation between fragmented state in a single trap and photonic cat state. And this was confirmed by direct mapping for special C_l distribution case. Relation between fragmented state and superposition of $|\beta\rangle$ and $|-\beta\rangle$ was specified by finding coefficient transformation between $|\mathrm{Even}\rangle, |\mathrm{Odd}\rangle$ basis and $|\beta\rangle, |-\beta\rangle$ basis. Very direct relation between $\Delta\rho_2(z, z')$ and quadrature fluctuation was shown significant still when only 5% of particles occupy additional mode, $\mathcal{F} = 0.1$.

In short, strong fluctuation of density-density correlation, usage of phase state formalism, and identification of fragmented state as photonic cat state are major findings dealt in this thesis. And these reveals interesting feature of fragmented state, in other word how exciting pictures many macroscopic modes and their correlation can generate. And analogy between fragmented state of cold atomic gas and superposition of coherent state in quantum optics is itself worthwhile and opened up possibility of new quantum macroscopic states. And one non trivial type of fragmentation, quantum many-body phenomenon, was revealed with opportunity. This fact supports that further unknown system with many macroscopic modes could possibly bear novel quantum many-body phenomenon.

There remains several future issues. Further investigation on stability of fragmented state issue can be done linear response theory against quantum depletion or parity breaking perturbation. Three-body recombination could key factor determining lifetime of fragmented state whether to be too short or long enough. Utilization of fragmented state, which is macroscopic superposition, for quantum metrology is probable. Single shot analysis can suggest whether we will really get only two types of single shot, of $|\beta\rangle$ or $|-\beta\rangle$, for single-trap fragmentation in real experiment. Since two 'cats' consisting fragmented state has *opposite* momentum in symmetric system, so this could be related to possible decoherence sources against fragmentation. And further analogy to photonic cat state e.g. find whether Wigner function in terms of $\hat{b}$ has negative probability region or not is also of interest. And, since fragmented stems from few macroscopic modes model, which is not exclusive for cold atomic gas, there can be more candidates or examples of fragmented state.

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Appendix A

Truncation of Field Operator

To see how a problem stated above happens and what should be done to avoid the problem, let us consider in general for which kind of operator $\hat{O}$ we can calculate by applying following truncation of field operator $\hat{\psi}(\vec{r})$

$$\hat{\psi}(\vec{r}) \tag{A.1}$$

to get $\left\langle \hat{O} \right\rangle$. To simplify an argument, we consider 'perfect' two-mod state $|\Psi\rangle$ such that

$$\langle\Psi|\, \hat{n}_i \,|\Psi\rangle = 0 \quad \text{for } i = M, M+1, \cdots \tag{A.2}$$

which leads to

$$\hat{a}_i \,|\Psi\rangle = 0, \;\; \langle\Psi|\, \hat{a}_i^\dagger = 0 \quad \text{for } i = M, M+1, \cdots . \tag{A.3}$$

This is also applied to mixed state of ensemble $\{|\Psi_n\rangle\}$ where

$$\left\langle \hat{n}_i \right\rangle = \sum_n P_n \,\langle\Psi_n|\, \hat{n}_i \,|\Psi_n\rangle = 0 \quad \text{for } i = M, M+1, \cdots . \tag{A.4}$$

with P_n which is probability in $n-th$ state $|\Psi_n\rangle$ and since

$$\langle\Psi_n|\, \hat{n}_i \,|\Psi_n\rangle \geq 0, \;\; P_n \geq 0 \tag{A.5}$$

we have

$$\hat{a}_i \,|\Psi_n\rangle = 0, \;\; \langle\Psi_n|\, \hat{a}_i^\dagger = 0 \quad \text{for } i = M, M+1, \cdots . \tag{A.6}$$

for all $|\Psi_n\rangle$ in ensemble.

In general, $\left\langle\hat{\psi}^\dagger(\vec{r})\hat{\psi}(\vec{r})\hat{\psi}^\dagger(\vec{r}')\hat{\psi}(\vec{r}')\right\rangle$ we can write as

$$\left\langle\hat{\psi}^\dagger(\vec{r})\hat{\psi}(\vec{r})\hat{\psi}^\dagger(\vec{r}')\hat{\psi}(\vec{r}')\right\rangle = \sum_{i,j,k,l=0}^{\infty} \psi_i^*(\vec{r})\psi_j(\vec{r})\psi_k^*(\vec{r}')\psi_l(\vec{r}')\langle\hat{a}_i^\dagger\hat{a}_j\hat{a}_k^\dagger\hat{a}_l\rangle \tag{A.7}$$

and immediately we see that if $i \geq M$ or $l \geq M$ then $\left\langle\hat{a}_i^\dagger\hat{a}_j\hat{a}_k^\dagger\hat{a}_l\right\rangle = 0$. Further, by giving normal ordering to $\left\langle\hat{a}_i^\dagger\hat{a}_j\hat{a}_k^\dagger\hat{a}_l\right\rangle$ we have

$$\left\langle\hat{a}_i^\dagger\hat{a}_j\hat{a}_k^\dagger\hat{a}_l\right\rangle = \left\langle\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_j\hat{a}_l\right\rangle + \left\langle\hat{a}_i^\dagger\left[\hat{a}_j,\hat{a}_k^\dagger\right]\hat{a}_l\right\rangle \tag{A.8}$$

therefore only $j = k$ survives for $j, k \geq M$ since $\hat{a}_j$ or $\hat{a}_k^\dagger$ will annihilate the state if $j \geq M$ or $k \geq M$. From this, we can also see that for normal ordered operator, it is enough to perform (A.1) to get M mode approximation because for normal ordered $\left\langle\hat{a}_i^\dagger\hat{a}_k^\dagger\hat{a}_j\hat{a}_l\right\rangle$ every i, j, k, l belongs to M mode configuration $(i, j, k, l < M)$ as long as (A.2) holds or is effectively suitable. Finally we get

$$\begin{aligned}&\left\langle\hat{\psi}^\dagger(\vec{r})\hat{\psi}(\vec{r})\hat{\psi}^\dagger(\vec{r}')\hat{\psi}(\vec{r}')\right\rangle \\ &= \sum_{i=0}^{M-1}\sum_{l=0}^{M-1}\left(\sum_{j=0}^{M-1}\sum_{k=0}^{M-1} + \sum_{j=M}^{\infty}\delta_{jk}\right)\psi_i^*(\vec{r})\psi_j(\vec{r})\psi_k^*(\vec{r}')\psi_l(\vec{r}')\langle\hat{a}_i^\dagger\hat{a}_j\hat{a}_k^\dagger\hat{a}_l\rangle.\end{aligned} \tag{A.9}$$

Now we know that second summation in bracket gives extra terms which we missed by directly applying (A.1) to $\left\langle\hat{\psi}^\dagger(\vec{r})\hat{\psi}(\vec{r})\hat{\psi}^\dagger(\vec{r}')\hat{\psi}(\vec{r}')\right\rangle$. In conclusion, to avoid such trouble we should perform truncation of filed operator after we express target operator $\hat{O}$ to be calculated as sum of normal ordered operators which comes from commuting every annihilation operators at the left of creation operators to the right of creation operators.

Appendix B

Time of Flight

Let's look at the TOF experiment result of $\left\langle\hat{\psi}^\dagger(z)\hat{\psi}^\dagger(z')\hat{\psi}(z')\hat{\psi}(z)\right\rangle$ based on non-interacting limit. This means that, turning off the trap at $t = 0$ and looking at time evolution of condensate(s). When interaction does not exist, time evolution can be simply described as propagation of constituent macroscopic wavefunctions while we are lying on occupation number representation. Let gravitational acceleration g to be 0 for now.

◇ Description of time evolution for non interacting many-body system
From [17], for the non interacting case one can choose to describe time evolution of the system as time evolution of the each macroscopic orbital $\psi_i(\vec{r}, t)$ as

$$i\partial_t\psi_i(\vec{r}, t) = \hat{h}\psi_i(\vec{r}, t) \tag{B.1}$$

where $\hat{h}$ is single particle Hamiltonian and every single particle density matrix(SPDM) and two particle density matrix(TPDM) remains invariant. Next, apply this picture to the eigenstate of harmonic oscillator since we will deal with BEC and NPC state by ground state and 1st excited state of harmonic oscillator.

◇ Time evolution of harmonic oscillator eigenstates ($g = 0$)

Every eigenstates of harmonic oscillator can be classified by angular frequency of trap ω and quantum number where mass m is fixed. j-th eigenstate

$\psi_j(z)$ is written as,

$$\psi_j(z) = \frac{1}{\sqrt{2^j\, j!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \mathrm{Exp}\left[-\frac{m\omega z^2}{2\hbar}\right] H_j\left(\sqrt{\frac{m\omega}{\hbar}}z\right), \qquad j = 0, 1, 2, \ldots. \tag{B.2}$$

Here H_j is Hermite function. Typical length scale w of eigenfunction is given as $w \equiv \sqrt{\hbar/m\omega}$. From [43, 44], j-th eigenstate $\psi_j^0(z) \equiv \psi_j(z, t = 0) \in \mathbb{R}$ after time t turning off the trap is,

$$\psi_j(z,t) = e^{i\phi(z,t)} \frac{e^{ij(\delta(t)+3\pi/2)}}{\sqrt{\omega t - i}} \psi_j^0(\tilde{z}) \tag{B.3}$$

Here $\delta(t)$, and $\tilde{z}$ are given as

$$\delta(t) = \arctan\left[\frac{1}{\omega t}\right], \ \tilde{z} = \frac{z}{\sqrt{1+\omega^2 t^2}} \tag{B.4}$$

And $\phi(z,t)$ is,

$$\phi(z,t) = \frac{m}{2\hbar t}(\tilde{z}\omega t)^2 - \frac{3\pi}{4} = \frac{m}{2\hbar t}\frac{\omega^2 t^2}{1+\omega^2 t^2}\ z^2 - \frac{3\pi}{4} \tag{B.5}$$

One can see that ω^{-1} determines a time scale for evolution. As $t \gg \omega^{-1}$, above variables are much simplified as

$$\delta(t) \to 0, \ \tilde{z} \to \frac{z}{\omega t}, \ \phi(z,t) \to \frac{m}{2\hbar t} z^2 - \frac{3\pi}{4} \tag{B.6}$$

In the following calculation, let ω to be a parameter which determines macroscopic orbitals describing each condensate. And absolute magnitude $|\psi_j(z,t)|^2$ is another thing to be pointed out here.

$$|\psi_j(z,t)|^2 = \frac{|\psi_j^0(\tilde{z})|^2}{\sqrt{1+\omega^2 t^2}} = \frac{1}{\sqrt{1+\omega^2 t^2}} \left|\psi_j^0\left(\frac{z}{\sqrt{1+\omega^2 t^2}}\right)\right|^2 \equiv |\psi_{jt}(z)|^2 \tag{B.7}$$

where $\psi_{jt}(z)$ is j-th eigenstate of oscillator which corresponds to rescaled frequency $\omega/(1+\omega^2 t^2)$, centered at $z = 0$ with the same normalization condition as $\psi_j^0(z)$. Thus absolute magnitude is expected to show only scaling behavior with recaled length $w(t)$ defined below.

$$\begin{aligned} w(t) =& \sqrt{\hbar\,/\,m\omega}\sqrt{1+\omega^2t^2} = w(t=0)\sqrt{1+\omega^2t^2} \\ \simeq& \; w(t=0)\,\omega t \quad \text{for} \quad t \gg \omega^{-1} \end{aligned} \tag{B.8}$$

Writing down ground state and 1st excited state of frequency ω which are our concern here,

$$\begin{aligned} \psi_0(z,t) =& e^{i\phi(z,t)}\frac{1}{\sqrt{\omega t - i}}\,\psi_0^0(\tilde{z}) \\ \psi_1(z,t) =& e^{i\phi(z,t)}\frac{e^{i(\delta(t)+3\pi/2)}}{\sqrt{\omega t - i}}\,\psi_1^0(\tilde{z}) \end{aligned} \tag{B.9}$$

The fact that ground state is not affected by $\delta(t)$, and 1st excited state is affected by $\delta(t)$ until $t \gg \omega^{-1}$ so $\delta(t) \to 0$ important thing to be noted. Also, look at $\psi_0^*(z,t)\psi_1(z,t)$

$$\psi_0^*(z,t)\psi_1(z,t) = \frac{e^{i(\delta(t)+3\pi/2)}}{\sqrt{1+\omega^2t^2}}\left(\psi_0^0(\tilde{z})\psi_0^1(\tilde{z})\right) = e^{i(\delta(t)+3\pi/2)}\left(\psi_{0t}(z)\psi_{1t}(z)\right) \tag{B.10}$$

at $t = 0$, $\delta(0) = \pi/2$ so there is no phase difference between $\psi_0(z,t)$ and $\psi_1(z,t)$ for the same position and time. But as $t \gg \omega^{-1}$, there happens $3\pi/2$ phase difference occur, and this makes a huge difference for correlation function when $t \gg \omega^{-1}$ from $t = 0$ correlation function.

◇ Relative phase evolution between even and odd wavefunction.

Suppose there are one even wavefunction $\psi_{even}^0(z)$ and one odd wavefunction $\psi_{odd}^0(z)$ at $t = 0$. Assuming that argument between $\psi_{even}^0(z)$ and $\psi_{odd}^0(z)$ is constant for all z, two functions can be treated as real function for every z at $t = 0$. Writing these two functions as linear combinations of each even and odd eigenfunctions of harmonic oscillator,

$$\begin{aligned} \psi_{even}^0(z) =& \sum_{j=0,2,\cdots} c_j\psi_j^0(z) \\ \psi_{odd}^0(z) =& \sum_{j=1,3,\cdots} c_j\psi_j^0(z) \end{aligned} \tag{B.11}$$

Every c_j is real number since $c_j = \int \left(\psi_{even}^0(z)\psi_j^0(z) + \psi_{odd}^0(z)\psi_j^0(z)\right) dz$, and

$\psi^0_{even}(z),\ \psi^0_{odd}(z)$ is real for all z. Now time evolution is obtained as,

$$\begin{aligned}\psi_{even}(z,t) &= \sum_{j=0,2,\cdots} c_j\psi_j(z,t) = \sum_{j=0,2,\cdots} c_j e^{i\phi(z,t)}\frac{e^{ij(\delta(t)+3\pi/2)}}{\sqrt{\omega t - i}}\psi^0_j(\tilde{z}) \\ \psi_{odd}(z,t) &= \sum_{j=1,3,\cdots} c_j\psi_j(z,t) = \sum_{j=1,3,\cdots} c_j e^{i\phi(z,t)}\frac{e^{ij(\delta(t)+3\pi/2)}}{\sqrt{\omega t - i}}\psi^0_j(\tilde{z})\end{aligned} \tag{B.12}$$

Argument between $\psi_{even}(z,t)$ and $\psi_{odd}(z,t)$ at time t is,

$$\begin{aligned}&\mathrm{Arg}\left[\psi_{even}(z,t),\ \psi_{odd}(z,t)\right] \\ &= \left[\sum_{j=0,2,\cdots} c_j e^{i\phi(z,t)}\frac{e^{ij(\delta(t)+3\pi/2)}}{\sqrt{\omega t - i}}\psi^0_j(\tilde{z}),\ \sum_{j=1,3,\cdots} c_j e^{i\phi(z,t)}\frac{e^{ij(\delta(t)+3\pi/2)}}{\sqrt{\omega t - i}}\psi^0_j(\tilde{z})\right] \\ &= \left[\sum_{j=0,2,\cdots} c_j\psi^0_j(\tilde{z})e^{ij(\delta(t)+3\pi/2)},\ \sum_{j=1,3,\cdots} c_j\psi^0_j(\tilde{z})e^{ij(\delta(t)+3\pi/2)}\right]\end{aligned} \tag{B.13}$$

As $t \gg \omega^{-1}$, $\delta(t) \to 0$ gives

$$\left[\left(c_0\psi^0_0(\tilde{z}) - c_2\psi^0_2(\tilde{z}) + \cdots\right),\ \left(-ic_1\psi^0_1(\tilde{z}) + ic_3\psi^0_3(\tilde{z}) - \cdots\right)\right] \tag{B.14}$$

Since every $c_j\psi^0_j(\tilde{z})$ is real, $\left(c_0\psi^0_0(\tilde{z}) - c_2\psi^0_2(\tilde{z}) + \cdots\right) \in \mathbb{R}$ and $\left(-ic_1\psi^0_1(\tilde{z}) + ic_3\psi^0_3(\tilde{z}) - \cdots\right) \in \mathbb{I}$. Thus argument between two wavefunctions is always $\pi/2$ or $3\pi/2$ else one of wavefunctions has 0 value. Here value of ω is about harmonic trap frequency correspondent to 'width' of wavefunctions.

Abstract in Korean

임계온도 이하의 희박한 원자기체는 계 전체에 걸친 큰 결맞음 등으로 흥미로운 물리 분야 중 하나이다. 이 저온 양자기체는 계의 변수들을 조정하기 용이하고, 외부의 원치 않는 간섭이 매우 적게 들어온다는 장점이 있다. 이런 점들이 하나의 거시적으로 점유된 모드(mode), 혹은 에너지 레벨, 를 이용해 보즈-아인슈타인 응집체(BEC)의 물리를 설명하고 묘사할 수 있게 한다.

본 논문에서, 파편화된 상태(fragmented state)-둘 혹은 더 많은 수의 거시적인 모드들이 존재하는 상태-를 다루었으며 둘 이상의 거시적인 모드들 간의 상관관계가 어떠한 흥미로운 효과를 가져올 수 있을지 살펴보았다. 그를 위해 파편화된 상태의 정의를 소개하고, 더 나아가 자명한 파편화된 상태인 이중 우물에서의 파편화된 상태를 걸러내기 위한 분류를 시도하였다. 파편화된 상태는, 절대 온도가 0으로 가는 극한에서 상호작용이 증가함에 따라 보즈-아인슈타인 응집체로부터 전이된 다체계 바닥 상태로써 나타날 것으로 생각된다.

거시적인 모드들은, 임계온도보다 높은 온도에 있는 이상기체들처럼 운동량 상태의 밀도행렬을 통해 단순하게 다루는 것이 불가능하다. 따라서 계에 대한 추가적인 제한이나 미리 알려진 특성 없이, 거시적인 모드들의 오비탈과 모드들의 점유수 등을 변분법적으로 계산하는 것은 수치적으로도 매우 힘든 일이다. 이 논문에서 다루는 다체계를 비균질 적인 하나의 덫 안에 있는 준-1차원 원자 기체로 한정하여, 거시적인 모드가 두 개인 경우$(1+1)$ 에 대해 다루려 하였다.

두 개의 거시적인 모드들로 구성된 유효 해밀토니안을 얻기 위해 장 연산자(field operator)를 두 번째 항까지 전개하고 잘라내었으며, 이는 그로스-피타예브스키 방정식에서 조금 더 나아가는 것이다. 이론을 준-1차원계의 함수로 얻기 위해 다른 방향들은 적분하였다. 그리고 원래 보즈-아인슈타인 응집체로 존재하는 거시적인 모드는 주로 가우시안이나 토마스-페르미 모양이며 짝수 패리티를 갖는다. 파편화된 상태는 상호작용을 제외한 에너지가 필연적으로 증가하므로, 추가되는

거시적인 모드를 홀수 패리티를 주어 덫 포텐셜의 영향을 줄이고 상호작용을 통해 에너지를 상대적으로 낮추기 위해 기존의 모드와 크게 겹치게 선택하였다.

이렇게 되는 경우, 제한적인 변분적 계산으로 에너지 방정식을 풀면, 상호작용의 세기가 올라감에 따라 파편화된 상태가 보즈-아인슈타인 응집체보다 더 낮은 에너지를 가지게 된다. 이 파편화된 상태는 상당한 크기의 음의 짝결맞음(pair coherence)를 가지고 있으며 이는 짝 터널링 항에 연관된다. 이 파편화된 상태는 거의 축퇴된 두 파편화된 상태들이 있으며, 작은 터널링 항에 의해 둘의 대칭 혹은 반대칭 중첩 상태 중 하나로 수렴하게 된다.

공간 결맞음 (spatial coherence)와 밀도-밀도 상관함수가 파편화된 상태의 고유의 특징을 잡아내는지에 대해 연구되었다. 밀도-밀도 상관함수의 강한 요동이 Time-of-Flight (TOF)이후에 존재하며, 이는 이중 우물에서 파편화된 상태의 Hanburry-Brown-Twiss (HBT) 상관관계와 비교되었다.

독립적인 두 보즈-아인슈타인 응집체들의 간섭 무늬 형성에 대해 좋은 해석을 준 위상 상태(phase state)를 이용해 단일 덫 안의 파편화된 상태에 대한 추가적인 해석을 하고자 하였고, 이를 일반적인 두개의 모드로 이루어진 상태에 대해 적용하기 위한 조건을 찾았다. 단일 덫 안의 파편화된 상태가 180도의 위상차이를 갖는 두 위상 상태의 중첩임을 알게 되었고, 파편화의 조건을 위상 상태를 이용해 표현함으로써 파편화된 상태와 특이한 상관함수간의 관계에 대해 조명하였다.

근사 결맞음 상태(approximate coherent state) 형식을 만들어, 파편화된 상태에 대한 추가적인 해석을 시도하였다. 위상 상태와 근사 결맞음 상태를 비교함으로써, 광자 고양이 상태와 파편화된 상태 간의 유사 관계에 대한 단서를 얻어 단일 덫 안의 파편화된 상태가 근사 결맞음 상태들의 중첩 상태, 즉 고양이 상태로 보여지는지에 대해 연구되었다. 일반적인 음의 짝결맞음 상태(NPC state)와 근사 결맞음 상태들의 중첩 상태간의 계수간의 관계를 찾아 유사 관계를 확립할 수 있었다. 더 나아가, 밀도-밀도 상관함수의 요동과 직교 위상 요동관의 직접적인 관계에 대해서도 다루었다.

주요어: 저온 보손 양자기체, 파편화, 단일 덫, 밀도-밀도 상관함수, 위상 상태, 근사 결맞음 상태, 음의 짝결맞음, 고양이 상태, 직교 위상 요동

학번: 2010-20356